

CHAMP: A CHEREDNIK ALGEBRA MAGMA PACKAGE

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ABSTRACT. We present a computer algebra package based on MAGMA for performing basic computations in rational Cherednik algebras at arbitrary parameters and in Verma modules for restricted rational Cherednik algebras. Part of this package is a new Las Vegas algorithm for computing the head and the constituents of a local module in characteristic zero. We used this method to compute for the first time the representation theory (including the Calogero–Moser families) for generic restricted rational Cherednik algebras for around half of the exceptional complex reflection groups. We furthermore apply it here to determine the representation theory of the restricted rational Cherednik algebra for the exceptional complex reflection group G_4 for all parameters. As a by-product this yields a proof of Martino’s conjecture for G_4 .

§1. Introduction

Based on the computer algebra system MAGMA we developed a package, called CHAMP, which provides an environment for performing basic computations in rational Cherednik algebras as introduced by Etingof–Ginzburg [7] and in Verma modules for restricted rational Cherednik algebras as introduced by Gordon [14]. It is freely available at <http://thielul.github.io/CHAMP/> and is designed to be highly flexible so that it is possible to work with arbitrary parameters (including indeterminates of a rational function field and thus covering the generic setting), with arbitrary reflection groups over arbitrary fields (including fields of positive characteristic as long as all reflections are diagonalizable), and with arbitrary realizations of the irreducible representations of the reflection groups (see §5). The development of this package was motivated by questions posed by Gordon [14, §7] (see 3.7) and by Martino’s conjecture [23] (see 3.10) which relates Calogero–Moser families with Rouquier families coming from Hecke algebras (see [5], [22], and [6]). For *exceptional* complex reflection groups not much was known about these questions and using CHAMP we could make significant progress (see §10).

CHAMP consists of around 15,000 lines of code at the moment. It contains a generic set of types and methods for (non-commutative) algebras with rewrite systems (see §2), and using the appropriate rules we are able to do basic computations in (restricted) rational Cherednik algebras. This can even be used to do computations in the more general Drinfeld–Hecke algebras (see [26] and [30]) which include the symplectic reflection algebras by Etingof–Ginzburg [7]. For the computation of Verma modules for restricted rational Cherednik algebras we devised quite efficient algorithms so that we can even construct and handle Verma modules of dimension around 3,000 (see §3). One of the central advances in CHAMP is a general Las Vegas algorithm for computing the head and the constituents of a local module over a field of characteristic zero (with some additional assumptions) to be described in §6. So far, there did not exist any efficient algorithm for solving this problem. Our idea is to use *finite field specializations* to transport the modules to an algebra over a finite field (see §4), then apply the MEATAXE, and use a method for *reconstructing* the head of the original module—the latter being the essential

Date: March 26, 2014

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part of our approach (see §5) and culminates in an algorithm we call MODFINDER. In total, this yields a Las Vegas algorithm, meaning that if it returns a result, it is the correct result but it can happen that the algorithm finishes without result. Despite the uncertainty in the success of this algorithm it turned out to be extremely efficient and successful for Verma modules of restricted rational Cherednik algebras. Using CHAMP we were able to compute for around half of the exceptional complex reflection groups the generic Calogero–Moser partitions (yielding proofs of Martino’s conjecture for all but one of these groups) and answer Gordon’s questions (see §10 and [30]). We note that the algorithms are so efficient that nearly all necessary computations can be performed on a single day on a home computer. In §9 we present the results for the smallest exceptional group G_4 for *all* parameters. This group is interesting in so far as it is the only exceptional group whose generic Calogero–Moser families are all singletons (see [1]) so that it was an open question for precisely which parameters they remain singletons and what they look like for arbitrary parameters. Our results yield as a by-product a full proof of Martino’s conjecture for G_4 .

We think that future applications of this package will enable us to better understand further problems about rational Cherednik algebras like the Calogero–Moser cell conjecture by Bonnafé–Rouquier [4].

Acknowledgements. I would like to thank Claus Fieker for showing me some tricks in MAGMA which led to improvements of CHAMP. I was partially supported by the *DFG Schwerpunktprogramm Darstellungstheorie 1388*.

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§2. Computing in rational Cherednik algebras

We postpone an explicit description of CHAMP to §7 and first lay down all necessary theoretical basics and abstractly describe the algorithms we conceived. We start by reviewing rational Cherednik algebras (see also [7], [14], [4], and [30]) in this paragraph and explain how they can be treated computationally. Instead of the complex numbers as base rings we consider a very general setup here to be able to treat generic parameters algebraically and to open the gate to analogous problems with modular reflection groups. We argue that the PBW theorem follows in this generality from the fact that there exists a terminating confluent rewrite system for rational Cherednik algebras. As a by-product, this formalizes an algorithm for computing in these algebras and proves its correctness.

2.1. Throughout, we fix a field K and a finite reflection group $\Gamma := (G, V)$ over K . This means that G is a non-trivial finite group, V is a finite-dimensional faithful KG -module, and G is generated by the set Ref_Γ of elements $s \in G$ which act as reflections on V , i.e., those elements whose fixed space $H_s := \text{Ker}(\text{id}_V - s)$ is of codimension one. By α_s^\vee we denote a *root* of s , i.e., a non-zero element of $\text{Im}(\text{id}_V - s)$, and by α_s we denote a *coroot* of s , i.e., an element of V^* whose kernel is equal to H_s . Both roots and coroots of reflections are unique up to scalars and our constructions will not depend on their choice.

We furthermore assume that all reflections in G are diagonalizable. This is equivalent to $\langle \alpha_s^\vee, \alpha_s \rangle \neq 0$ for all $s \in \text{Ref}_\Gamma$. As all reflections in Γ are of finite order, this is certainly satisfied if Γ is *non-modular*, i.e., the characteristic of K is coprime to the order of G . In the modular case the general orthogonal groups in their natural representation in case K is of characteristic not equal to 2, the symmetric group S_n in the representation attached to the partition $(n-1, 1)$ in case K is of characteristic not equal to 2 and coprime to n , and some modular reductions of exceptional complex reflection groups satisfy this property for example (see [30]).

2.2. In addition to Γ we furthermore fix a commutative K -algebra R , an element $t \in R$, and a map $c : \mathcal{C}_\Gamma \rightarrow R$ from the set \mathcal{C}_Γ of conjugacy classes of reflections of Γ to R . The *rational Cherednik algebra* of Γ in (t, c) is defined as the quotient $H_{t,c}$ of $R\langle V \oplus V^* \rangle \rtimes RG$ by the ideal $I_{t,c}$ generated by the relations

$$(1) \quad [x, x'] = 0 \quad \text{for all } x, x' \in V^*,$$

$$(2) \quad [y, y'] = 0 \quad \text{for all } y, y' \in V,$$

and

$$(3) \quad [y, x] = t\langle y, x \rangle + \sum_{s \in \text{Ref}_\Gamma} \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} c(s)s \quad \text{for all } x \in V^*, y \in V.$$

Here, we denote by $R\langle V \rangle$ the tensor algebra of V^* over R and by $R[V]$ we denote the symmetric algebra of V^* , i.e., the quotient of $R\langle V \rangle$ by the ideal generated by the elements $xx' - x'x$ for $x, x' \in V^*$. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between V and V^* and $R\langle V \oplus V^* \rangle \rtimes RG$ denotes the semi-direct product of the tensor algebra of $V^* \oplus V$ over R and the group algebra over R . As we assumed that all reflections are diagonalizable, we have $\langle \alpha_s^\vee, \alpha_s \rangle \neq 0$ so that the last relation is always well-defined. Note that it is also independent of the choice of the roots and coroots.

2.3. Let $\mathbf{y} := (y_i)_{i=1}^n$ be a basis of V with dual basis $\mathbf{x} := (x_i)_{i=1}^n$. A basis of $R\langle V \rangle$ is formed by the elements of the form $\mathbf{x}_\alpha := \prod_{i \in \mathbb{N}} x_{\alpha_i}$ with $\alpha \in [1, n]_0^{\mathbb{N}}$, where $[1, n]_0^{\mathbb{N}}$ denotes the set of functions $[1, n] \rightarrow \mathbb{N}$ such that all but finitely many values are equal to zero and we set $x_0 := 1$. Moreover, a basis of $R[V]$ is formed by the elements $\mathbf{x}^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ with $\alpha \in \mathbb{N}^n$. The choice of a basis provides us with a natural R -linear section of the quotient morphism $R\langle V \rangle \twoheadrightarrow R[V]$ by mapping $\mathbf{x}^\alpha \in R[V]$ to $\prod_{i=1}^n x_i^{\alpha_i} \in R\langle V \rangle$. In the same way we have a natural R -linear section of $R\langle V^* \rangle \twoheadrightarrow R[V^*]$. As an R -module the semi-direct product $R\langle V \oplus V^* \rangle \rtimes RG$ is isomorphic to $R\langle V \oplus V^* \rangle \otimes_R RG$, and in the context of rational Cherednik algebras we prefer to use that this is in turn isomorphic to $R\langle V \rangle \otimes_R RG \otimes_R R\langle V^* \rangle$. The two sections above can thus be put together to yield an R -linear section $s_{\mathbf{y}}$ of the quotient morphism $R\langle V \oplus V^* \rangle \rtimes RG \twoheadrightarrow R[V \oplus V^*] \rtimes RG$. The image $N_{\mathbf{y}}$ of $s_{\mathbf{y}}$ is the free R -submodule of $R\langle V \oplus V^* \rangle \rtimes RG$ with basis $\mathbf{x}^\alpha g \mathbf{y}^\beta$ and we get a commutative diagram

$$\begin{array}{ccc}
& & N_{\mathbf{y}} \\
& \nearrow s_{\mathbf{y}} & \downarrow \\
& R\langle V \oplus V^* \rangle \rtimes RG & \\
\swarrow & & \searrow \\
R[V \oplus V^*] \rtimes RG & \xrightarrow{\pi} & H_{t,c}
\end{array}$$

where the dashed arrows are morphisms of R -modules only and π is the composition of $s_{\mathbf{y}}$ with the quotient morphism. This morphism is actually independent of the choice of \mathbf{y} and is called the *PBW morphism*. It is clear from the relations (1) and (2) that π is surjective so that the elements $\mathbf{x}^\alpha \mathbf{g} \mathbf{y}^\beta$ generate $H_{t,c}$ as an R -module. The essence of the *PBW theorem* for rational Cherednik algebras is that π is in fact an isomorphism (equivalently, the restriction of the quotient morphism $R\langle V \oplus V^* \rangle \rtimes RG \twoheadrightarrow H_{t,c}$ to $N_{\mathbf{y}}$ is injective for one, and then any, basis \mathbf{y}). Hence, the elements $\mathbf{x}^\alpha \mathbf{g} \mathbf{y}^\beta$ with $\alpha, \beta \in \mathbb{N}^n$ form an R -basis of $H_{t,c}$. We call such a basis a *PBW basis*.

The PBW theorem was originally proven by Etingof–Ginzburg [7] in the case $K = R = \mathbb{C}$. Their proof, however, seems to be not easily extendable to our general setting. Ram–Shepler [26] instead gave a proof in the same case which is formalized and extended in [30]. The advantage of this approach is not only that it can be adapted to give a proof of the PBW theorem over general base rings but that it also provides the theoretical foundation of our computational approach to rational Cherednik algebras. To explain this let us first formalize the role of $N_{\mathbf{y}}$ in the PBW theorem.

2.4. Definition. Let A be an algebra over a commutative ring R and let $I \trianglelefteq A$ be an ideal. A *weak normal form* of A/I is an R -submodule $N \subseteq A$ such that any element of A is modulo I equivalent to an element of N , i.e., the restriction $\pi|_N$ of the quotient morphism $\pi : A \twoheadrightarrow A/I$ to N is still surjective. For $a \in A$ we call the elements in $\mathcal{N}_N(a) := \pi|_N^{-1}(\pi(a)) = \pi^{-1}(\pi(a)) \cap N$ the *normal forms* of a with respect to N , and similarly we define $\mathcal{N}_N(\bar{a}) := \pi|_N^{-1}(\bar{a}) = \pi^{-1}(\bar{a}) \cap N$ for $\bar{a} \in A/I$. If every element of A has a unique normal form with respect to N , i.e., $\pi|_N : N \twoheadrightarrow A/I$ is an isomorphism of R -modules, we say that N is a *normal form* of A/I .

Finding a normal form for a quotient of (commutative) polynomial ring by an ideal is one of the central problems of computational commutative algebra and it can be solved via Gröbner bases as explained in the following example.

2.5. Example. Let $A := K[X]$ be the polynomial ring over a field K in the variables $\mathbf{X} := (X_i)_{i=1}^n$. Let \prec be a monomial order on A . Let $I \trianglelefteq A$ be an ideal and let $G := \{g_1, \dots, g_s\}$ be a Gröbner basis of I with respect to \prec , i.e., $\text{LT}(I) = \text{LT}(G)$, where $\text{LT}(-)$ denotes the ideal generated by the leading terms. Let

$$C(I) := \{\mathbf{X}^\alpha \mid \alpha \in \mathbb{N}^n \text{ and } \mathbf{X}^\alpha \text{ is not divisible by some } \text{LT}(g) \text{ for } g \in G\} \subseteq A.$$

Then $N_I := \langle C(I) \rangle_K \subseteq A$ is a normal form of A/I (see [9, §1.2]).

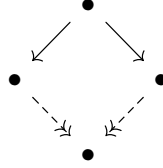
We can reformulate the PBW theorem as stating that the R -submodule $N_{\mathbf{y}}$ is a normal form for $H_{t,c} = (R\langle V \oplus V^* \rangle \rtimes RG)/I_{t,c}$. We will show this by proving that there exists a terminating confluent rewrite system having $N_{\mathbf{y}}$ as the set of normal forms. To this end, let us first recall some basic notions around rewrite systems (see [3]).

2.6. Definition. A *rewrite system* is a pair $\mathcal{A} := (A, \rightarrow)$ consisting of a set A and a binary relation \rightarrow on A . We write $a \rightarrow b$ if $(a, b) \in \rightarrow$. This relation is called the *rewrite relation* of

\mathcal{A} . The reflexive-transitive closure of \rightarrow is denoted by \twoheadrightarrow . An element $a \in A$ is *reducible* if there is some $b \in A$ with $a \neq b$ and $a \rightarrow b$. Otherwise it is called *irreducible* (or in *normal form*). A *normal form* of an element $a \in A$ is an irreducible element $b \in A$ with $a \twoheadrightarrow b$. We denote by $\mathcal{N}_{\mathcal{A}}(a)$ the set of normal forms of a . The rewrite system \mathcal{A} is (*uniquely*) *normalizing* if every element $a \in A$ has a (unique) normal form. It is called *terminating* if there does not exist an infinite chain $a_1 \rightarrow a_2 \rightarrow \dots$. It is called *locally confluent* if

$$\forall a, b, c \in A (c \leftarrow a \rightarrow b \Rightarrow \exists d \in A (c \twoheadrightarrow d \leftarrow b)) .$$

This condition is precisely the commutativity of the diagram



where the vertices denote the corresponding elements of A and the dashed arrows indicate the existence condition. Finally, \mathcal{A} is called *confluent* if

$$\forall a, b, c \in A (c \leftarrow a \twoheadrightarrow b \Rightarrow \exists d \in A (c \twoheadrightarrow d \leftarrow b)) .$$

Very helpful for proving confluence of a rewrite system is Newman's lemma which states that a terminating rewrite system is confluent if and only if it is locally confluent (see [3, Theorem 1.2.1]). Let us record some further elementary facts about rewrite systems.

2.7. Lemma. The following holds for a rewrite system $\mathcal{A} := (A, \rightarrow)$:

- (a) If \mathcal{A} is terminating, then \mathcal{A} is normalizing.
- (b) If \mathcal{A} is confluent, then any element of A has at most one normal form.
- (c) \mathcal{A} is uniquely normalizing if and only if it is normalizing and confluent.

Proof. Assertions (a) and (b) are easy to see. If \mathcal{A} is uniquely normalizing, it is normalizing by definition. To see that \mathcal{A} is confluent let $a, b, c \in A$ with $c \leftarrow a \twoheadrightarrow b$. Let \tilde{c} be a normal form of c and let \tilde{b} be a normal form of b . We then have $a \twoheadrightarrow b \twoheadrightarrow \tilde{b}$ and $a \twoheadrightarrow c \twoheadrightarrow \tilde{c}$. Since \tilde{b} and \tilde{c} are irreducible, they are both normal forms of a . But then $\tilde{b} = \tilde{a} = \tilde{c}$, where \tilde{a} is the unique normal form of a . This shows that \mathcal{A} is confluent. The other direction is evident. ■

We are targeting to establish a rewrite system on an algebra with respect to an ideal. Such a rewrite system should satisfy some natural compatibility conditions. We propose the following definition (there seems to be no established general theory yet).

2.8. Definition. Let A be an algebra over a commutative ring R and let $I \trianglelefteq A$ be an ideal. A *rewrite system* for A/I is a rewrite system $\mathcal{A} := (A, \rightarrow)$ on A satisfying the following properties:

- (a) If $a \rightarrow b$, then $a \equiv b \pmod{I}$ for all $a, b \in A$.
- (b) If $a \in A$ is irreducible, also ra is irreducible for all $r \in R$.
- (c) If $a, b \in A$ are irreducible, also $a + b$ is irreducible.

We can now relate the two notions of normal forms in 2.4 and 2.6. The following two lemmas are the key to the PBW theorem.

2.9. Lemma. Let A be an algebra over a commutative ring R , let $I \trianglelefteq A$ be an ideal and let $\mathcal{A} := (A, \rightarrow)$ be a rewrite system for A/I . The following holds:

- (a) If $a \twoheadrightarrow b$, then $a \equiv b \pmod{I}$ for all $a, b \in A$.

(b) If \mathcal{A} is normalizing, then

$$N_{\mathcal{A}} := \bigcup_{a \in A} \mathcal{N}_{\mathcal{A}}(a) \subseteq A$$

is a weak normal form of A/I with $\mathcal{N}_{\mathcal{A}}(a) \subseteq \mathcal{N}_{N_{\mathcal{A}}}(a)$ for all $a \in A$.

Proof. The first assertion follows immediately from 2.8(a) and the fact that \equiv is both reflexive and transitive. Furthermore, 2.8(b) and 2.8(c) imply that $N_{\mathcal{A}}$ is an R -submodule of A and it is then a weak normal form of A/I due to (a) \blacksquare

2.10. Lemma. Let A be an algebra over a commutative ring R , let $I \trianglelefteq A$ be an ideal and let $\mathcal{A} := (A, \rightarrow)$ be a normalizing rewrite system for A/I . The following are equivalent:

- (a) $N_{\mathcal{A}}$ is a normal form of A/I .
- (b) $a \rightarrow 0$ for all $a \in I$.

In this case \mathcal{A} is uniquely normalizing and $\mathcal{N}_{\mathcal{A}}(a) = \mathcal{N}_{N_{\mathcal{A}}}(a)$ for all $a \in A$.

Proof. Suppose that $N_{\mathcal{A}}$ is a normal form of A/I . Then $\mathcal{N}_{N_{\mathcal{A}}}(a)$ is a singleton for all $a \in A$. Since \mathcal{A} is normalizing and $\mathcal{N}_{\mathcal{A}}(a) \subseteq \mathcal{N}_{N_{\mathcal{A}}}(a)$, this implies that $\mathcal{N}_{\mathcal{A}}(a) = \mathcal{N}_{N_{\mathcal{A}}}(a)$ and so $\mathcal{N}_{\mathcal{A}}(a)$ is also a singleton. Hence, \mathcal{A} is uniquely normalizing. Moreover, if $a \in I$, then $\mathcal{N}_{\mathcal{A}}(a) = \mathcal{N}_{N_{\mathcal{A}}}(a) = \pi^{-1}(\pi(a)) \cap N_{\mathcal{A}} = \pi^{-1}(0) \cap N_{\mathcal{A}} = I \cap N_{\mathcal{A}} = \{0\}$. Hence, $a \rightarrow 0$ for all $a \in I$.

Now, suppose that (b) holds. To show that $N_{\mathcal{A}}$ is a normal form, we show that the restriction $\pi|_{N_{\mathcal{A}}}$ of the quotient morphism $\pi : A \twoheadrightarrow A/I$ to $N_{\mathcal{A}}$ is injective. If \tilde{a} is an element of the kernel of this morphism, then $\tilde{a} \in N_{\mathcal{A}} \cap I$, so \tilde{a} is an irreducible element contained in I . But the assumption that $a \rightarrow 0$ for all $a \in I$ implies that whenever $a \in I$ is irreducible, then already $a = 0$. Hence $\tilde{a} = 0$ and so $N_{\mathcal{A}}$ is a normal form of A/I . \blacksquare

2.11. Remark. If \mathcal{A} is normalizing and satisfies $a \rightarrow 0$ for all $a \in I$, then it follows from 2.10 and 2.7(c) that \mathcal{A} is confluent. The condition $a \rightarrow 0$ for all $a \in I$ might, however, be stronger than confluence. In other words, confluence of \mathcal{A} alone might not be sufficient for making $N_{\mathcal{A}}$ into a normal form for A/I .

2.12. Remark. Defining rewrite relations for A/I is much more intricate than it seems at first—in particular when it comes to verifying confluence and the property $a \rightarrow 0$ for all $a \in I$. Usually, one would tend to define rewrite relations on *symbolic monomials* of A , which we understand as symbolic concatenations of elements of A symbolizing a product, and then extend these relations to *symbolic expressions*, i.e., symbolic monomials involving parentheses, addition and subtraction symbols. But this approach leads to the following major issue. Let $a \in A$ be an irreducible element and let $b \in A$ be a reducible element. In A we have of course $a = a + b - b$ but as symbolic expressions a and $a + b - b$ are distinct. Since b is reducible and we extended the rewrite rules by linearity, also $a + b - b$ is reducible. This is a contradiction since in A this symbolic term becomes equal to a which is irreducible. Because of this one has to be very careful when defining rewrite relations for A/I . We can avoid this problem by defining rewrite rules on basis elements of A and then extending these linearly. We formalize this in the following definition.

2.13. Definition. Let $\mathbf{a} := (a_{\lambda})_{\lambda \in \Lambda}$ be an R -basis of A . In this context we call the elements a_{λ} also *monomials* of A and by *terms* we understand multiples ra_{λ} with $r \in R \setminus \{0\}$. If $a \in A$ we say that a term ra_{λ} is a *term of a* if $ra_{\lambda} \neq 0$ occurs in the basis representation of a . Now, suppose that \rightarrow is a subset of $(a_{\lambda})_{\lambda \in \Lambda'} \times A$ for some subset $\Lambda' \subseteq \Lambda$, i.e., \rightarrow relates some monomials of A with elements of A . We extend \rightarrow to a relation \rightarrow' as follows:

- (a) If $a \in A$ and ra_{λ} is a term of a with $a_{\lambda} \rightarrow b$, then $a \rightarrow' a - ra_{\lambda} + rb$.
- (b) If $a_{\lambda} \rightarrow b$ and $a_{\mu} = xa_{\lambda}y$ for some $\lambda, \mu \in \Lambda$ and $x, y \in A$, then $a_{\mu} \rightarrow' xby$.

The first extension rule should be understood as removing the term ra_λ from a and replacing it by b . The second extension rule means that we can apply rules to “submonomials” of monomials. We call the rules defined by \rightarrow the *elementary rules* of the resulting rewrite system and call rewrite systems defined like this *monomial rewrite systems*.

2.14. It is easy to see that a monomial rewrite system on an algebra A satisfies properties 2.8(b) and 2.8(c). So, what remains to be verified to establish it as a rewrite system for A/I is property 2.8(a) on elementary rules (note that I is a two-sided ideal). Suppose that in this case we can furthermore show that the resulting rewrite system \mathcal{A} for A/I is terminating and that $a \rightarrow 0$ for all $a \in I$ holds. Then we know from 2.7(a) that \mathcal{A} is normalizing and so it follows from 2.10 that \mathcal{A} is already uniquely normalizing. Furthermore, the module theoretic notion of normal forms in 2.4 coincides with the rewrite system theoretic one in 2.6.

2.15. We now define a monomial rewrite system $\mathcal{A}_{t,c,y}$ on $R\langle V \otimes V^* \rangle \rtimes RG$ with respect to the R -basis $x_\alpha g y_\beta$ by the following elementary rules:

- $$\begin{aligned}
 (4) \quad & x_j x_i \rightarrow x_i x_j && \text{for } j > i, \\
 (5) \quad & y_j y_i \rightarrow y_i y_j && \text{for } j > i, \\
 (6) \quad & y_i x_j \rightarrow x_j y_i + t\langle y_i, x_j \rangle + \sum_{s \in \text{Ref}_\Gamma} \frac{\langle y_i, \alpha_s \rangle \langle \alpha_s^\vee, x_j \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} c(s) s && \text{for all } i, j.
 \end{aligned}$$

A tedious but straightforward computation (see [30, §16]) shows that $\mathcal{A}_{t,c,y}$ is a terminating rewrite system for the quotient $H_{t,c}$ with $a \rightarrow 0$ for all $a \in I_{t,c}$. It is obvious that $N_{\mathcal{A}_{t,c,y}} = N_y$ and as 2.10 implies that N_y is a normal form for $H_{t,c}$, this proves the PBW theorem.

2.16. Our approach to the PBW theorem using rewrite systems also gives us an algorithm for computing in rational Cherednik algebras. The only remaining problem is to realize the “covering algebra” $R\langle V \otimes V^* \rangle \rtimes RG$ in the computer. We propose the following solution. We choose a generating set $\mathbf{g} := (g_i)_{i=1}^r$ of G and an injective map $w : G \hookrightarrow W(\mathbf{g})$ into the set of words over the alphabet \mathbf{g} . Computationally, this is for example possible by computing a rewrite system for G with respect to the generators \mathbf{g} via the Knuth–Bendix algorithm. This can easily be done in MAGMA and in this way we also choose such a map in CHAMP. Now, instead of working with the rewrite system $\mathcal{A}_{t,c,y}$ on $R\langle V \oplus V^* \rangle \rtimes RG$ we work with a rewrite system $\mathcal{A}'_{t,c,y}$ on the free algebra $R\langle \mathbf{x} \cup \mathbf{g} \cup \mathbf{y} \rangle$ having the analogous elementary rules (4) to (6) and the additional elementary rules

- $$\begin{aligned}
 (7) \quad & g_k x_i \rightarrow {}^{g_k} x_i g_k \quad \text{for all } i \text{ and } k, \\
 (8) \quad & y_i g_k \rightarrow g_k {}^{g_k^{-1}} y_i \quad \text{for all } i \text{ and } k,
 \end{aligned}$$

where ${}^g y$ for $g \in G$ and $y \in V$ denotes the action of g on y and ${}^g x$ for $x \in V^*$ denotes the (dual) action of g on x . This yields again a terminating confluent rewrite system but the normal forms it produces are not yet the same as those of $\mathcal{A}_{t,c,y}$ as we still have to take care of the relations we have to impose on $R\langle \mathbf{g} \rangle$ to get RG . To this end, we apply a “final normalization” to the group algebra part of the terms of the normal form produced by $\mathcal{A}'_{t,c,y}$ by mapping it to G , compute its image under w and replace the group algebra part by the corresponding expression in $R\langle \mathbf{g} \rangle$. This yields an algorithm for computing in rational Cherednik algebras (see algorithm 1).

2.17. Remark. The first part of algorithm 1 (lines 1 to 16) is just the straightforward way to implement a monomial rewrite system in a computer and in this way we also do it in CHAMP (see §7). We realize the search for an applicable rule (line 7) by numbering the generators, representing a monomial as a corresponding integer sequence, and then search if a sequence describing an initial monomial of the elementary rewrite rules occurs as a subsequence. This elementary part is actually the one which consumes most time in our implementation. The last

Algorithm 1: Computing in rational Cherednik algebras

Data: An element $a = \sum_{\substack{\alpha, \beta \in [1, n]_0^{\mathbb{N}} \\ \gamma \in G_0^{\mathbb{N}}}} r_{\alpha, \beta, \gamma} \mathbf{x}^\alpha \mathbf{g}_\gamma \mathbf{y}^\beta \in R\langle \mathbf{x} \cup \mathbf{g} \cup \mathbf{y} \rangle$

Result: The representation of a in the PBW basis of $H_{t, c}$ with respect to \mathbf{y}

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1 badterms := the terms of  $a$ ;
2 goodterms :=  $\emptyset$ ;
3 while badterms  $\neq \emptyset$  do
4   choose  $t \in \text{badterms}$ ;
5   badterms := badterms  $\setminus \{t\}$ ;
6    $c :=$  the coefficient of  $t$ ;  $m :=$  the monomial of  $t$ ;
7   Check if one of the monomials on the left hand side of the elementary rules (4) to (8)
   occurs in  $m$ , i.e., check if  $m = m_1 m_2 m_3$  with  $m_2$  one of these monomials;
8   if this is true then
9     Let  $m_2 \rightarrow b$  be such a rule;
10    Compute  $t_{\text{rew}} := c m_1 b m_3$  in  $R\langle \mathbf{x} \cup \mathbf{g} \cup \mathbf{y} \rangle$ ;
11    badterms := badterms  $\cup$  the terms of  $t_{\text{rew}}$ ;
12  else
13    goodterms := goodterms  $\cup \{t\}$ ;
14  end
15 end
16  $a_{\text{rew}} := \sum_{t \in \text{goodterms}} t = \sum_{\substack{\alpha, \beta \in [1, n]_0^{\mathbb{N}} \\ \gamma \in G_0^{\mathbb{N}}}} r'_{\alpha, \beta, \gamma} \mathbf{x}^\alpha \mathbf{g}_\gamma \mathbf{y}^\beta \in R\langle \mathbf{x} \cup \mathbf{g} \cup \mathbf{y} \rangle$ ;
17  $\tilde{a} := 0$ ;
18 for  $\alpha, \beta, \gamma$  as above with  $r'_{\alpha, \beta, \gamma} \neq 0$  do
19    $w_\gamma :=$  the image under  $w$  of  $\mathbf{g}_\gamma$  evaluated in  $G$ , and interpreted again in  $R\langle \mathbf{g} \rangle$ ;
20    $\tilde{a} := \tilde{a} + r'_{\alpha, \beta, \gamma} \mathbf{x}^\alpha w_\gamma \mathbf{y}^\beta$ ;
21 end
22 return  $\tilde{a}$ ;
```

part of the algorithm (starting from line 17) is the “final normalization” discussed above. We point out that the rewrite process can be highly recursive since with each rule we apply to a term, new terms might be created which have to be brought to normal form. The fact that we are allowed to choose any applicable rule in this algorithm (line 9) is due to the confluence of the rewrite system. Although confluence is a great property, we have seen in experiments that different choices of applicable rules have an impact on the size of the “normalization tree”, i.e., on the number of rules we have to apply to get a normal form. It is absolutely unclear yet if there is a formal optimal strategy for choosing rules, minimizing the total number of rules which have to be applied to arrive at the normal form.

2.18. Remark. The proof of the PBW theorem is given [30, §16] by the same arguments for the much more general Drinfeld–Hecke algebras (see also [26]). The class of such algebras include for example the symplectic reflection algebras by Etingof–Ginzburg [7]. With the same algorithm 1 we can thus compute in these algebras, too (see §7).

§3. Computing in restricted rational Cherednik algebras and in Verma modules

Besides the capability of performing basic computations in rational Cherednik algebras it is one aim of CHAMP to compute representation theoretic properties of *restricted* rational Cherednik algebras. These algebras—which were first seriously studied by Gordon [14]—are

finite-dimensional quotients of $H_{0,c}$ by a centrally generated ideal and they possess (partially established, partially conjectural) relations to Hecke algebras. These relations are one reason for studying (restricted) rational Cherednik algebras. In this section, we will review the basic properties of these algebras, explain what representation theoretic problems we are interested in, and address some computational issues. We include a quick review of Martino's conjecture to be very precise about what we computed and to ensure that these computations yield proofs of this conjecture in the cases under consideration.

3.1. The \mathbb{N} -graded ring

$$Z_\Gamma := K[V]^G \otimes_K K[V^*]^G \subseteq K[V \oplus V^*]^G$$

of *bi-invariants* maps under the PBW morphism into the center of $H_{0,c}$ and embeds the scalar extension Z_Γ^R as a central subalgebra of $H_{0,c}$. For $K = R = \mathbb{C}$ this was proven by Etingof–Ginzburg [7] and Gordon's proof [14] in this case also works without modifications in our general setting. We can thus view $H_{0,c}$ as a Z_Γ^R -algebra. Note that since R is a flat K -module, the scalar extension Z_Γ^R is simply given by replacing K by R above. As the extension $K[V]^G \subseteq K[V]$ is finite (see [16, 12.27]), the PBW theorem implies that $H_{0,c}$ is a finite Z_Γ^R -module. The finiteness implies (see [30, §6, §17]) that we have a decomposition

$$\text{Simp}(H_{0,c}) = \coprod_{\mathfrak{m} \in \text{Max}(Z_\Gamma^R)} \text{Simp}(H_{0,c}(\mathfrak{m}))$$

of the sets of simple modules, where Max denotes the maximal ideal spectrum and $H_{0,c}(\mathfrak{m}) := H_{0,c}/\mathfrak{m}H_{0,c}$ is the *specialization* of $H_{0,c}$ in $\mathfrak{m} \in \text{Max}(Z_\Gamma^R)$. This decomposition follows essentially from the fact that maximal ideals and left primitive ideals coincide in PI rings. The advantage is that on the right hand side we have finite-dimensional algebras over fields which might be easier to study than $H_{0,c}$ itself.

There is one maximal ideal of Z_Γ^R which is defined independently of Γ , namely the augmentation ideal

$$\mathfrak{m}_{0,0} := (Z_\Gamma^R)_+ = (R[V]_+^G \otimes_R R[V^*]_+^G) + (R[V]_+^G \otimes_R R[V^*]_+^G).$$

The specialization $\overline{H}_c := H_{0,c}(\mathfrak{m}_{0,0})$ is called the *restricted rational Cherednik algebra* of Γ in c . Its simple modules are precisely the simple $H_{0,c}$ -modules annihilated by $\mathfrak{m}_{0,0}H_{0,c}$. Recall that the *coinvariant algebra* $K[V]_G$ of Γ is the quotient of $K[V]$ by the Hilbert ideal, which is the ideal in $K[V]$ generated by the augmentation ideal $K[V]_+^G$ of $K[V]^G$. It follows at once from the PBW theorem that the PBW morphism induces an R -module isomorphism

$$(9) \quad R[V]_G \otimes_R R[G] \otimes_R R[V^*]_G \cong \overline{H}_c,$$

implying that \overline{H}_c is a free R -module with

$$\dim_R \overline{H}_c = \dim_R R[V]_G \cdot |G| \cdot \dim_R R[V^*]_G.$$

In case both $K[V]^G$ and $K[V^*]^G$ are polynomial (this holds for example in the non-modular setting by a theorem by Bourbaki–Chevalley–Serre as Γ is a reflection group), the extensions $K[V]^G \subseteq K[V]$ and $K[V^*]^G \subseteq K[V^*]$ are free of dimension equal to $|G|$. This implies that in this case $\dim_R \overline{H}_c = |G|^3 = \dim_{Z_\Gamma^R} H_{0,c}$.

3.2. Let us now choose a Gröbner basis of the Hilbert ideal of Γ with respect to some monomial order. As in example 2.5 this allows us to compute a monomial basis $(\bar{\mathbf{x}}^\lambda)_{\lambda \in \Lambda}$ of the coinvariant algebra $K[V]_G$, where $\Lambda \subseteq \mathbb{N}^n$ is some finite subset and $\bar{\mathbf{x}} := (\bar{x}_i)_{i=1}^n$ are the images of the $x_i \in K[V]$ in $K[V]_G$. Similarly, we obtain a monomial basis $(\bar{\mathbf{y}}^\sigma)_{\sigma \in \Sigma}$ of $K[V^*]_G$. Then by the above \overline{H}_c is a free R -module with basis $(\bar{\mathbf{x}}^\lambda g \bar{\mathbf{y}}^\sigma)_{\lambda \in \Lambda, \sigma \in \Sigma, g \in G}$ and we call a basis of this form a *PBW basis* of \overline{H}_c .

3.3. Algorithm 1 can easily be extended to do computations in \overline{H}_c , i.e., to compute their PBW basis representations. Again we work in the algebra $R\langle \mathbf{x} \cup \mathbf{g} \cup \mathbf{y} \rangle$. The output of algorithm 1 is

an element of the form $\tilde{a} = \sum_{\alpha, \beta \in \mathbb{N}^n, \gamma \in G_0^{\mathbb{N}}} r_{\alpha, \beta, \gamma} \mathbf{x}^\alpha \mathbf{g}_\gamma \mathbf{y}^\beta$. To obtain the PBW representation of the image of this element in \overline{H}_c we have to apply two further “final normalizations”, namely we have to apply the relations coming from the Hilbert ideals. To this end, we just loop over each term $r_{\alpha, \beta, \gamma} \mathbf{x}^\alpha \mathbf{g}_\gamma \mathbf{y}^\beta$, compute the representation in the basis $(\bar{\mathbf{x}}^\lambda)_{\lambda \in \Lambda}$ of the image $\bar{\mathbf{x}}^\alpha$ of \mathbf{x}^α in $K[V]_G$, lift back this representation using the natural linear section $\bar{x}_i \mapsto x_i$ of the quotient morphism $K[V] \twoheadrightarrow K[V]_G$, and replace \mathbf{x}^α by this lift. Of course we have to do the analogous operation with \mathbf{y}^β . The computation of the basis representation of $\bar{\mathbf{x}}^\alpha$ is performed by the rewrite system defined by the Gröbner basis and is computationally well established. In precisely this way we made CHAMP capable of computing in restricted rational Cherednik algebras (see §7).

3.4. Now, we turn our attention to representation theoretic problems of \overline{H}_c which are originally due to Gordon [14]. First of all, note that $R\langle V \oplus V^* \rangle \rtimes RG$ is naturally a \mathbb{Z} -graded R -algebra by putting V^* in degree 1, G in degree 0, and V in degree -1 . The elements in (1) to (3) defining the ideal $I_{0,c}$ are all homogeneous so that $H_{0,c}$ inherits this \mathbb{Z} -grading. Since the Hilbert ideals are homogeneous, it follows moreover that the restricted rational Cherednik algebra \overline{H}_c also inherits this \mathbb{Z} -grading.

3.5. Gordon [14] observed that the triangular decomposition (9) of \overline{H}_c governs its representation theory by employing a general theory of Holmes–Nakano [17]. The key tool is the *Verma functor*

$$\Delta_c := \overline{H}_c \otimes_{\overline{H}_{c,r}} q_{c,r*}(-) : \overline{H}_{c,m}(\text{gr})\text{mod} \rightarrow \overline{H}_c(\text{gr})\text{mod}$$

between categories of finitely generated (graded) modules. Here, $\overline{H}_{c,m}$ and $\overline{H}_{c,r}$ denote the subalgebras of degree zero and non-positive degree elements of \overline{H}_c with respect to the \mathbb{Z} -grading, respectively (these are the middle part and right part of the triangular decomposition, respectively). The PBW morphism induces isomorphisms $\overline{H}_{c,m} \cong RG$ and $\overline{H}_{c,r} \cong RG \ltimes R[V^*]$ which are in fact isomorphisms of R -algebras. Mapping elements of V to zero yields a surjective algebra morphism $q_{c,r} : \overline{H}_{c,r} \twoheadrightarrow \overline{H}_{c,m}$ and by $q_{c,r*}$ we denote the induced inflation functor $\overline{H}_{c,m}(\text{gr})\text{mod} \rightarrow \overline{H}_{c,r}(\text{gr})\text{mod}$. It is not hard to see that

$$(10) \quad \Delta_c(W) \cong R[V] \otimes_R W$$

as R -modules provided that W is free as an R -module (see [17] or [30, §18]).

3.6. Now, suppose that R is a field and that KG splits (the latter holds for example if K is of characteristic zero by a theorem by Benard [2]). Although Holmes–Nakano [17] assume for their theory an algebraically closed base field, their arguments also work when the algebra is just split (see [30, §18]) and show that for each simple KG -module λ the corresponding Verma module $\Delta_c(\lambda) := \Delta_c(\lambda^R)$ of \overline{H}_c is an indecomposable module with simple head $L_c(\lambda)$ and that $(L_c(\lambda))_{\lambda \in \text{Simp}(KG)}$ is a system of representatives of the simple \overline{H}_c -modules. As a Verma module $\Delta_c(\lambda)$ is graded, it follows from [15, Proposition 3.5] that its radical is a graded submodule so that $L_c(\lambda)$ is graded. Arguments by Bonnafé–Rouquier [4, Proposition 9.2.5] furthermore show that \overline{H}_c itself splits. There is now a natural correspondence between simple KG -modules and simple \overline{H}_c -modules and so the distribution of simple \overline{H}_c -modules into the blocks of \overline{H}_c yields a partition CM_c of the set of simple KG -modules whose members are called the *Calogero–Moser c -families*.

3.7. Gordon formulated in [14, §7] the following problems concerning the representation theory of \overline{H}_c :

- (a) Find the graded G -character of the simple modules $L_c(\lambda)$. This includes knowing their dimensions and their Poincaré series.
- (b) Determine the composition factors of the Verma modules $\Delta_c(\lambda)$.
- (c) Determine the Calogero–Moser c -families.

These problems were originally formulated in case $K = R = \mathbb{C}$ (which essentially covers the situation of base fields of characteristic zero). We cannot go into details about what is already known about these problems in this case (see [7, 16.2, 16.4], [8], [14, 6.4, 7.3], [1, §3.3], [23], [24], and [30]). The point is that almost nothing is known for *exceptional* complex reflection groups and this was one reason for the development of CHAMP.

3.8. The above problems are formulated for any point c of the affine K -scheme $\mathfrak{R}_\Gamma := \mathbb{A}_K^{\#\mathcal{C}_\Gamma}$, which is the parameter space of restricted rational Cherednik algebras. This infinite amount of parameters would be a serious issue for a computational approach but the following two facts allow us to reduce this to finitely many problems. First of all, it is proven in [29] that decomposition morphisms are generically trivial. This means essentially that once we know the solution to 3.7(a) and 3.7(b) for the generic point c of \mathfrak{R}_Γ , i.e., c is the family of indeterminates of the rational function field $K((c_s)_{s \in \mathcal{C}_\Gamma})$, then we know the solution for all c in a dense open subset of \mathfrak{R}_Γ . This *generic situation* is really the starting point of computational considerations and is supported by CHAMP. Similarly, it is proven in [4] (see also [30, §11]) that blocks show the same behavior, meaning that once we know the generic Calogero–Moser families CM_c , we know them for all c in a dense open subset of \mathfrak{R}_Γ . After the generic situation is understood, we have to determine the locus of “exceptional parameters” and continue the above process. This is exactly what we will do exemplarily for the exceptional group G_4 in §9.

3.9. Before we discuss our approach to the computational solution of these problems, let us first explain why the Calogero–Moser families are interesting. To this end, we need a different type of parameters for rational Cherednik algebras due to Ginzburg–Guay–Opdam–Rouquier [13]. Let \mathcal{A}_Γ be the set of G -orbits of reflection hyperplanes of Γ . For a reflection hyperplane H of Γ the stabilizer subgroup G_H is cyclic of some order e_H prime to the characteristic of K . This order is constant along the G -orbit Ω of H so that we can denote it by e_Ω . We denote by Ω_Γ the set of pairs (Ω, j) with $\Omega \in \mathcal{A}_\Gamma$ and $1 \leq j \leq e_\Omega - 1$, and denote by $\Omega_\Gamma^\#$ the set of pairs (Ω, j) with $0 \leq j \leq e_\Omega - 1$. Let $\overline{\mathfrak{R}}_\Gamma$ be the affine K -scheme $\mathbb{A}_K^{\#\Omega_\Gamma^\#}$. For $k \in \overline{\mathfrak{R}}_\Gamma(R)$ we now define a function $c_k : \mathcal{C}_\Gamma \rightarrow R$ by

$$(11) \quad c_k(s) := \sum_{j=0}^{e_{\Omega_s}-1} \det(s)^j (k_{\Omega_s, j+1} - k_{\Omega_s, j}) ,$$

where Ω_s is the G -orbit of the reflection hyperplane of s and we consider the index j always modulo e_{Ω_s} . We set $H_{0,k} := H_{0,c_k}$. It is not hard to see that (11) yields a surjective R -linear map $\Phi_\Gamma(R) : \overline{\mathfrak{R}}_\Gamma(R) \rightarrow \mathfrak{R}_\Gamma(R)$, and that this defines a surjective K -scheme morphism $\Phi_\Gamma : \overline{\mathfrak{R}}_\Gamma \rightarrow \mathfrak{R}_\Gamma$. We can thus think of $\overline{\mathfrak{R}}_\Gamma$ as an artificial extension of the parameter space for restricted rational Cherednik algebras of Γ . On $\overline{\mathfrak{R}}_\Gamma$ we define an involution $(\cdot)^\sharp$ by $k^\sharp := (k_{\Omega, -j})$. The closed subscheme $\overline{\mathfrak{R}}_\Gamma^0$ of $\overline{\mathfrak{R}}_\Gamma$ consisting of all k with $k_{\Omega, 0} = 0$ is stable under this involution and we call its points *Cherednik parameters of GGOR type* for Γ . Note that Φ_Γ restricts to an isomorphism between $\overline{\mathfrak{R}}_\Gamma^0$ and \mathfrak{R}_Γ so that this can be considered as a re-parametrization of \mathfrak{R}_Γ .

Now, assume that K is of characteristic zero and that Γ is irreducible. Chlouveraki’s [6] essential hyperplanes define a union $\overline{\mathcal{E}}_\Gamma$ of hyperplanes in $\overline{\mathfrak{R}}_\Gamma$ defined by integral equations and attached to any point $k \in \overline{\mathfrak{R}}_\Gamma$ is a partition Rou_k of the simple KG -modules whose members are called the *Rouquier k -families*. We cannot go into details about Rouquier families here (see [5], [22], [6], and [4]) and just note how we can define them for a general base field K of characteristic zero instead of just $K = \mathbb{C}$. To this end, we have to choose a realization Γ' of Γ over the complex numbers, which is possible as Γ admits a realization over its character field. When doing this we have to keep track of the orbits of hyperplanes of reflections to avoid changing the parameters. Then Chlouveraki’s theory defines the essential hyperplanes in $\overline{\mathfrak{R}}_{\Gamma'}$ and the Rouquier k -families for any $k \in \overline{\mathfrak{R}}_{\Gamma'}(\mathbb{C})$. The point is now that these families are already uniquely determined by the essential hyperplanes k lies on. This and the fact that

the essential hyperplanes are defined by integral equations allows us transport the essential hyperplanes to $\overline{\mathfrak{R}}_\Gamma$ and to define Rouquier families for any point of $\overline{\mathfrak{R}}_\Gamma$. We remark that for the definition of Rouquier families we tacitly assume the validity of some standard assumptions about Hecke algebras (see [6, 4.2.3]) which are not known to hold for all complex reflection groups. The interest in Calogero–Moser families is now justified by the following conjecture.

3.10. Conjecture (Martino, [23]). Assume that K is of characteristic zero and that Γ is irreducible. The following holds:

- (a) Rou_{k^\sharp} is a refinement of $\text{CM}_k := \text{CM}_{c_k}$ for any $k \in \overline{\mathfrak{R}}_\Gamma$.
- (b) There is a non-empty open subset U of $\overline{\mathfrak{R}}_\Gamma$ such that $\text{Rou}_{k^\sharp} = \text{CM}_k$ for all $k \in U$.

3.11. We call the first part of the conjecture the *special parameter conjecture* and the second part the *generic parameter conjecture*. Because restricted rational Cherednik algebras and cyclotomic Hecke algebras always split, it is enough to consider some particular realization of each type of complex reflection groups in the Shephard–Todd classification and then prove the conjecture for K -points. Furthermore, we note that for $k \in \overline{\mathfrak{R}}_\Gamma$ the k -cyclotomic Hecke algebra is naturally isomorphic to the k^0 -cyclotomic Hecke algebra, where k^0 is obtained from k by setting $k_{\Omega,0}$ to zero for all Ω (this follows from [4, 2.1.13]). Hence, we can equivalently consider the conjecture just for points of $\overline{\mathfrak{R}}_\Gamma^0$ as originally formulated by Martino. Due to the behavior of Calogero–Moser families explained in 3.8 and due to the behavior of Rouquier families explained in 3.9 the generic parameter conjecture is equivalent to $\text{Rou}_k = \text{CM}_k$, where k is the generic point of $\overline{\mathfrak{R}}_\Gamma^0$.

3.12. Martino’s conjecture is known to be true for symmetric and imprimitive complex reflection groups by [23], [1], and [24]. The generic parameter conjecture is known to be true for G_4 by [1], and also for G_5 , G_6 , G_8 , G_{10} , G_{23} , G_{24} by [28]. It was shown in [28], however, that the generic parameter conjecture fails for G_{25} . In all cases where this conjecture is known to hold it was proven by determining the Calogero–Moser families and comparing them to the Rouquier families which have been determined by Chlouveraki [6]. So far, there is no theoretical explanation for this connection, and the counter-example in case G_{25} makes it even harder to understand the situation.

Bonnafé–Rouquier [4] have, however, pointed out a neat argument why there could exist such a connection at all. First of all, the *Euler element* of $H_{0,c}$, introduced in [13], is defined as

$$(12) \quad \text{eu}_c = \sum_{i=1}^n y_i x_i + \sum_{s \in \text{Ref}_\Gamma} \frac{1}{\varepsilon_s - 1} c(s) s = \sum_{i=1}^n x_i y_i + \frac{\varepsilon_s}{\varepsilon_s - 1} c(s) s ,$$

where ε_s denotes the non-trivial eigenvalue of s . The definition does not depend on the choice of a basis \mathbf{y} of V with dual basis \mathbf{x} . This element is known to be central and its image in \overline{H}_c is again a non-trivial central element. Let Ω_λ^c be the central character of the simple \overline{H}_c -module $L_c(\lambda)$. Then the values of these characters on the Euler element yield a partition Eu_c of the simple KG -modules which is finer than CM_c . We call its members the *Euler c -families*. It is proven in [4, 10.2.2] that for $k \in \overline{\mathfrak{R}}_\Gamma$ the equality $\Omega_\lambda^k(\text{eu}_k) = c_\lambda(k^\sharp)$ holds, where $c_\lambda(k^\sharp)$ is a constant multiple of the “ q -logarithm” of the value of the central character of the simple module belonging to λ of the k^\sharp -cyclotomic Hecke algebra on the central element π coming from the center of the braid group of Γ (see [4, §2.2]). These values define similarly a partition Π_{k^\sharp} of the simple KG -modules which is finer than Rou_{k^\sharp} . We thus have

$$(13) \quad \text{CM}_k \leq \text{Eu}_k = \Pi_{k^\sharp} \geq \text{Rou}_{k^\sharp} ,$$

where \leq denotes refinement. Of course, this does not explain why $\text{CM}_k \geq \text{Rou}_{k^\sharp}$ should hold.

3.13. Next to the Euler families there is another type of families giving a further approximation of the Calogero–Moser families. Namely, for a fixed simple KG -module λ we collect all

constituents of $\Delta_c(\lambda)$. For each of these constituents S_μ we again collect all constituents of $\Delta_c(\mu)$ etc. This process stabilizes and gives us a partition Ver_c of the simple KG -modules whose members we call the *Verma c -families*. As Verma modules are indecomposable, these are always contained in a family coming from a block of \overline{H}_c , i.e., each Verma family is contained in a Calogero–Moser family, so $\text{Ver}_c \leq \text{CM}_c$. We thus get a tower

$$(14) \quad \text{Ver}_c \leq \text{CM}_c \leq \text{Eu}_c$$

giving us approximations of CM_c from two sides. The Euler families are easily computable using the characters of the simple KG -modules (see [4] or [28]). The Verma families in turn can be computed in many cases by the methods we will discuss in the next paragraphs. Usually, the above tower collapsed, i.e., the Verma families were equal to the Euler families and thus equal to the Calogero–Moser families.

3.14. Remark. Recently, Bonnafé–Rouquier [4, §13.4] have proven that in case K is of characteristic zero, the Verma families are in fact equal to the Calogero–Moser families (we observed this property before in our explicit computations). This is now the theoretical foundation showing that the key to determining the Calogero–Moser families are the Verma families.

3.15. After stating the main problems we are interested in, let us go back to computational issues. Clearly, we first have to find an explicit description of the Verma modules for any computational approach to these problems, so let us start with this. Let $\rho : G \rightarrow \text{End}_K(W)$ be a finite-dimensional K -representation. Then the Verma module $\Delta_c(\rho)$ is uniquely determined by the action of the generators $\mathbf{x} \cup \mathbf{g} \cup \mathbf{y}$ of \overline{H}_c , and as $\Delta_c(\rho)$ is free and finitely generated as an R -module, these actions are described by some matrices. In this way a Verma module can be represented in the computer once we have chosen bases and understood the action. To this end, we choose besides a basis $\mathbf{y} := (y_i)_{i=1}^n$ of V with dual basis $\mathbf{x} := (x_i)_{i=1}^n$ and a generating system $\mathbf{g} := (g_i)_{i=1}^r$ of G also a monomial basis $\bar{\mathbf{x}}^\Lambda := (\bar{\mathbf{x}}^\lambda)_{\lambda \in \Lambda}$ of $K[V]_G$ as described in 3.2. Furthermore, we fix a basis $\mathbf{w} := (w_k)_{k=1}^d$ of W . Then an R -basis of $\Delta_c(\rho) \cong R[V]_G \otimes_R W$ is formed by the elements $\bar{\mathbf{x}}^\lambda \otimes w_k$, and with respect to this basis we now describe the action of the generators.

First, let us consider the action of x_i on $\Delta_c(\rho)$. We have

$$x_i.(\bar{\mathbf{x}}^\mu \otimes w_k) = (x_i \bar{\mathbf{x}}^\mu) \otimes w_k.$$

Hence, if the basis representation of $x_i \bar{\mathbf{x}}^\mu \in K[V]_G$ in the basis $\bar{\mathbf{x}}^\Lambda$ is

$$x_i \bar{\mathbf{x}}^\mu = \sum_{\lambda \in \Lambda} \alpha_\lambda^{i,\mu} \bar{\mathbf{x}}^\lambda,$$

then

$$(15) \quad x_i.(\bar{\mathbf{x}}^\mu \otimes w_k) = \sum_{\lambda \in \Lambda} \alpha_\lambda^{i,\mu} \bar{\mathbf{x}}^\lambda \otimes w_k$$

is the basis representation of $x_i.(\bar{\mathbf{x}}^\mu \otimes w_k) \in \Delta_c(\rho)$ in the basis $\bar{\mathbf{x}}^\Lambda \otimes \mathbf{w}$. So, we actually just need to understand the action of the x_i on the coinvariant algebra $K[V]_G$ and this can computationally be solved using Gröbner bases.

Now, let us consider the action of g_i on $\Delta_c(\rho)$. We have

$$g_i.(\bar{\mathbf{x}}^\mu \otimes w_k) = (g_i \bar{\mathbf{x}}^\mu) \otimes w_k = ({}^{g_i} \bar{\mathbf{x}}^\mu g_i) \otimes w_k = {}^{g_i} \bar{\mathbf{x}}^\mu \otimes {}^{g_i} w_k.$$

Hence, if the basis representation of ${}^{g_i} \bar{\mathbf{x}}^\mu \in K[V]_G$ in the basis $\bar{\mathbf{x}}^\Lambda$ is

$${}^{g_i} \bar{\mathbf{x}}^\mu = \sum_{\lambda \in \Lambda} \beta_\lambda^{i,\mu} \bar{\mathbf{x}}^\lambda$$

and the basis representation of ${}^{g_i} w_k$ in the basis \mathbf{w} is

$${}^{g_i} w_k = \sum_{t=1}^d \gamma_t^{i,k} w_t,$$

then the basis representation of $g_i \cdot (\bar{\mathbf{x}}^\mu \otimes w_k)$ in the basis $\bar{\mathbf{x}}^\Lambda \otimes \mathbf{w}$ is

$$(16) \quad g_i \cdot (\bar{\mathbf{x}}^\mu \otimes w_k) = \left(\sum_{\lambda \in \Lambda} \beta_\lambda^{i, \mu} \bar{\mathbf{x}}^\lambda \right) \otimes \left(\sum_{t=1}^d \gamma_t^{i, k} w_t \right) = \sum_{\lambda \in \Lambda} \sum_{t=1}^d \beta_\lambda^{i, k} \gamma_t^{i, k} \bar{\mathbf{x}}^\lambda \otimes w_t.$$

So, to understand the action of g_i in $\Delta_c(\rho)$ we need to understand the action of g_i on the coinvariant algebra $K[V]_G$ and on the KG -module W . The first can again be computationally achieved using Gröbner bases, the second is no problem when we have an explicit realization of ρ .

Now, we come to the hardest part, namely the action of y_i on $\Delta_c(\rho)$. This is the point where the structure of the restricted rational Cherednik algebra enters the game. Namely, to write the element $y_i(\bar{\mathbf{x}}^\mu \otimes w_k) = (y_i \bar{\mathbf{x}}^\mu) \otimes w_k$ in the basis $\bar{\mathbf{x}}^\Lambda \otimes \mathbf{w}$, we first have to rewrite $y_i \bar{\mathbf{x}}^\mu$ in the PBW-basis of \bar{H}_c . A straightforward proof by induction shows that in $H_{0,c}$ the equation

$$(17) \quad [y_i, \mathbf{x}^\mu] = \sum_{t=1}^n \sum_{s \in \text{Ref}_\Gamma} \sum_{l=0}^{\mu_t-1} c(s) (y_i, x_t)_s x_1^{\mu_1} \cdots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \cdots ({}^s x_n)^{\mu_n} s$$

holds, where $(y_i, x_t)_s := \frac{\langle y_i, \alpha_s \rangle \langle \alpha_s^\vee, x_t \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle}$. Using this formula we get

$$\begin{aligned} y_i(\bar{\mathbf{x}}^\mu \otimes w_k) &= (y_i \bar{\mathbf{x}}^\mu) \otimes w_k = (\bar{\mathbf{x}}^\mu y_i + [y_i, \bar{\mathbf{x}}^\mu]) \otimes w_k = (\bar{\mathbf{x}}^\mu y_i) \otimes w_k + [y_i, \bar{\mathbf{x}}^\mu] \otimes w_k \\ &= \sum_{t=1}^n \sum_{s \in \text{Ref}_\Gamma} \sum_{l=0}^{\mu_t-1} c(s) (y_i, x_t)_s x_1^{\mu_1} \cdots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \cdots ({}^s x_n)^{\mu_n} s \otimes w_k \\ &= \sum_{t=1}^n \sum_{s \in \text{Ref}_\Gamma} \sum_{l=0}^{\mu_t-1} c(s) (y_i, x_t)_s x_1^{\mu_1} \cdots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \cdots ({}^s x_n)^{\mu_n} \otimes {}^s w_k \\ (18) \quad &= \sum_{s \in \text{Ref}_\Gamma} \sum_{t=1}^n \sum_{l=0}^{\mu_t-1} c(s) (y_i, x_t)_s x_1^{\mu_1} \cdots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \cdots ({}^s x_n)^{\mu_n} \otimes {}^s w_k, \end{aligned}$$

where we used that $(\bar{\mathbf{x}}^\mu y_i) \otimes w_k = 0$ by definition of $\Delta_c(\rho)$. This expression is not yet a basis expression in the basis $\bar{\mathbf{x}}^\Lambda \otimes \mathbf{w}$ but by rewriting the elements on the left hand side of the tensor products in the basis $\bar{\mathbf{x}}^\Lambda$ as above using Gröbner bases and rewriting the elements on the right hand side in the basis \mathbf{w} immediately gives a basis expression. Hence, with the formulas in (15), (16), and (18) we can explicitly compute the Verma module $\Delta_c(\rho)$ and represent it in this way in a computer. Note, however, that it still needs an explicit method—like Gröbner bases—to rewrite elements in the coinvariant algebra in a chosen (monomial) basis.

3.16. Some parts of formula (18) occur multiple times. In particular if one wants to consecutively compute Verma modules for different KG -representations, one can split off these parts to increase efficiency. We propose the following approach. Fix $i \in [1, n]$, $s \in \text{Ref}_\Gamma$, and $\mu \in \Lambda$. Let $X_\mu^{(i,s)} = (X_{\mu,\eta}^{(i,s)})_{\eta \in \Lambda}$ be such that $X_{\mu,\eta}^{(i,s)}$ is the coefficient of $\bar{\mathbf{x}}^\eta$ in the basis representation of

$$\sum_{t=1}^n \sum_{l=0}^{\mu_t-1} (y_i, x_t)_s x_1^{\mu_1} \cdots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \cdots ({}^s x_n)^{\mu_n} \in K[V]_G$$

in the basis $\bar{\mathbf{x}}^\Lambda$. We can consider $X_\mu^{(i,s)}$ as a row vector and by varying μ we get a matrix $X^{(i,s)} \in \text{Mat}_{\Lambda \times \Lambda}(K)$ satisfying

$$(19) \quad y_i(\bar{\mathbf{x}}^\mu \otimes w_k) = \sum_{s \in \text{Ref}_\Gamma} c(s) \sum_{\eta \in \Lambda} X_{\mu,\eta}^{(i,s)} \bar{\mathbf{x}}^\eta \otimes {}^s w_k.$$

Note that the matrix $X^{(i,s)}$ is independent of the representation ρ and even of c so that it can be used again for further computations. For the computation of $X^{(i,s)}$ we can define for fixed

$\mu \in \Lambda$ the following two expressions, indexed by $t \in [1, n]$:

$$(20) \quad p_\mu^{\text{start}}(t) := x_1^{\mu_1} \cdots x_{t-1}^{\mu_{t-1}},$$

$$(21) \quad p_\mu^{\text{end}}(t) := x_{t+1}^{\mu_{t+1}} \cdots x_n^{\mu_n}.$$

Then for $s \in \text{Ref}_\Gamma$ the row vector $X_\mu^{(i,s)}$ can be determined by computing the basis representation of the element

$$(22) \quad \sum_{s \in \text{Ref}_\Gamma} \sum_{t=1}^n (y_i, x_t)_s p_\mu^{\text{start}}(t) \left(\sum_{l=0}^{\mu_t-1} x_t^l (s x_t)^{\mu_t-l-1} \right) s p_\mu^{\text{end}}(t).$$

The above methods for computing $\Delta_c(\rho)$ are implemented in exactly this way in CHAMP. To use the grading of Verma modules we implemented a new type `ModGr` allowing to handle graded modules in general. Even Verma modules of dimension a few thousand can be computed quite fast in this way and as we allow sparse matrices in CHAMP, the memory usage is usually still very modest.

After this initial problem being solved, we turn to the actual questions, namely: how can we compute the simple modules $L_c(\lambda)$, i.e., the heads of the Verma modules, and how can we compute the constituents of the Verma modules? Over finite fields, this can be achieved using the MEATAXE algorithm (see [25], [20], [18], [19, §7.4], [21, §1.3], [10, 7.1.1]), which is also implemented in MAGMA. In the generic situations (where the base ring is a rational function field) and in case of base rings of characteristic zero, however, there does not exist any algorithm capable of solving our problems. Although there are some recent approaches to a “characteristic zero MEATAXE”—so for example the general method developed by Steel [27], which is also implemented in MAGMA—no existing algorithm was successful even in smaller examples (see the experiments in 8.2 proving this). We therefore conceived a method aiming to solve this problem. Although our whole idea is based on necessary conditions so that the resulting algorithm might not produce a result at all, it turned out to be amazingly successful and efficient for Verma modules of restricted rational Cherednik algebras and was the key tool of our progress on Gordon’s questions for exceptional complex reflection groups (see §10).

Our approach is very general and relies on the fact that we can solve the problems over finite fields using the MEATAXE. As we do not have a finite field at hand, we first need a way to transfer the situation to a finite field and then we have to figure out what the situation over the finite field tells us about our original situation. The following proposition—formulated abstractly—is the main ingredient for our approach.

3.17. Definition. If \mathcal{A} and \mathcal{B} are two essentially small abelian categories, then a group morphism $d : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$ of the zeroth K-groups is called *positive* if $d(K_0^+(\mathcal{A})) \subseteq K_0^+(\mathcal{B})$, where K^+ is the submonoid represented by objects, and it is called *strongly positive* if it is positive and $d([X]) = 0$ implies $[X] = 0$ for all $[X] \in K_0^+(\mathcal{A})$.

3.18. Proposition. Let \mathcal{A} and \mathcal{B} be two abelian categories of finite length and let $d : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$ be a strongly positive morphism. Let $X \in \mathcal{A}$. The following holds:

- (a) If $d([X])$ is simple, then X itself is simple.
- (b) Let $(S_i)_{i \in I}$ be a set of representatives of the simple objects of \mathcal{A} and let $(T_j)_{j \in J}$ be a set of representatives of the simple \mathcal{B} -objects. Let $X \in \mathcal{A}$ and let $J_X^d := \{j \in J \mid [d([X])] : T_j] \neq 0\}$. Suppose that there exists a subset $I_X^d \subseteq I$ such that $[X : S_i] = 0$ for all $i \in I \setminus I_X^d$ and such that there exists a bijection $\lambda_X^d : J_X^d \rightarrow I_X^d$ with $d([S_{\lambda_X^d(j)}]) = T_j$ for all $j \in J_X^d$. Then

$$[X] = \sum_{j \in J_X^d} [d([X]) : T_j] [S_{\lambda_X^d(j)}].$$

and in this case we say that d is X -generic.

Proof.

(a) Suppose that X is not simple. Then we can write $[X] = [X_1] + [X_2]$ with $[X_1], [X_2] \neq 0$ and we get the relation $[T] = d([X]) = d([X_1]) + d([X_2])$ in $K_0^+(\mathcal{B})$ with $T \in \mathcal{B}$ simple. Since d is strongly positive, we have $d([X_1]), d([X_2]) \neq 0$. But then the above relation in $K_0^+(\mathcal{B})$ is impossible. Hence, X must be simple.

(b) The basis representation of $[X]$ is

$$[X] = \sum_{i \in I} [X : S_i][S_i] = \sum_{i \in I_X^d} [X : S_i][S_i].$$

Using the fact that λ_X^d is a bijection, we get

$$d([X]) = \sum_{i \in I_X^d} [X : S_i]d([S_i]) = \sum_{j \in J_X^d} [X : S_{\lambda_X^d(j)}]d([S_{\lambda_X^d(j)}]) = \sum_{j \in J_X^d} [X : S_{\lambda_X^d(j)}][T_j].$$

Since the basis representation of $d([X])$ is

$$d([X]) = \sum_{j \in J} [d([X]) : T_j][T_j] = \sum_{j \in J_X^d} [d([X]) : T_j][T_j],$$

the claim is proven. ■

Applying this to the Grothendieck groups of finite-dimensional algebras A and B over fields shows us that if we have a strongly positive morphism $d : G_0(A) \rightarrow G_0(B)$ and we can compute decompositions in $G_0(B)$ —for example if the base field of B is finite using the MEATAXE!—, then we can computationally prove that an A -module is simple and we have a chance of computing decompositions of A -modules in $G_0(A)$. The morphism d is really the link between a computationally manageable ring B and the ring A . Our proposition leads us to the following two ideas.

3.19. For computing the head of a finite-dimensional local module V over a finite-dimensional algebra A over a field we propose the following method:

- (a) Find a strongly positive morphism $d : G_0(A) \rightarrow G_0(B)$ with B a finite-dimensional algebra over a *finite* field.
- (b) Create a submodule J of V , which is to be considered as a candidate for the radical, compute the quotient V/J and check using the MEATAXE if $d(V/J)$ is irreducible. If it is, then we know that V/J is simple and is therefore the head of V .

3.20. Let A be a finite-dimensional algebra over a field. Suppose that we have a family $(V_\lambda)_{\lambda \in \Lambda}$ of finite-dimensional local A -modules with heads $(S_\lambda)_{\lambda \in \Lambda}$. Suppose furthermore that this family is *constituent-closed*, meaning that every constituent of a member V_λ of this family is the head S_μ of some V_μ . We then have

$$[V_\lambda] = \sum_{\mu \in \Lambda} m_{\lambda, \mu} [S_\mu] \in G_0(A)$$

for some $m_{\lambda, \mu} \in \mathbb{N}$ and we propose the following method for computing these decomposition numbers:

- (a) Find a strongly positive morphism $d : G_0(A) \rightarrow G_0(B)$ with B a finite-dimensional algebra over a *finite* field such that $d(S_\lambda)$ is simple for all $\lambda \in \Lambda$.
- (b) For each $\lambda \in \Lambda$ compute using the MEATAXE the constituents $(T_{\lambda, \theta})_{\theta \in \Theta_\lambda}$ and their multiplicities $m_{\lambda, \theta}$. Now, check if there exists an injection $\iota_\lambda : \Theta_\lambda \hookrightarrow \Lambda$ such $d(S_\mu) \cong T_{\lambda, \theta}$ for some $\mu \in \Lambda$ and $\theta \in \Theta_\lambda$ if and only if $\mu = \iota_\lambda(\theta)$. In this case

$$[V_\lambda] = \sum_{\mu \in \Lambda} m_{\lambda, \mu} [S_\mu] \in G_0(A),$$

where $m_{\lambda, \iota_\lambda(\theta)} := m_{\lambda, \theta}$ for $\theta \in \Theta_\lambda$ and $m_{\lambda, \mu} := 0$ for all $\mu \notin \text{Im } \iota_\lambda$.

While decomposition morphisms—more precisely, compositions of decomposition morphisms which do not necessarily have to be decomposition morphisms themselves, whence the formulation using strongly positive morphisms—will certainly play a central role for finding appropriate strongly positive morphisms to algebras over finite fields, it is completely unclear at this stage what we should do in 3.19(b) to produce a candidate for the radical of a local module. In the following two sections we will discuss methods to solve these two problems. Our final algorithm is presented in §6.

§4. Finite field specializations of restricted rational Cherednik algebras

Let us fix a Dedekind domain \mathcal{O} with quotient field K , a normal K -algebra R , and an R -algebra A which is free and finitely generated as an R -module.

4.1. Definition. A *finite field specialization* of A is a pair (\mathfrak{m}, u) such that:

- (a) \mathfrak{m} is a maximal ideal of \mathcal{O} with finite residue field.
- (b) u is a K -point of the K -scheme $\mathrm{Spec}(R)$ such that the K -algebra $A(u) := u^*A$ splits and has an $\mathcal{O}_{\mathfrak{m}}$ -free $\mathcal{O}_{\mathfrak{m}}$ -structure $\tilde{A}(u)$, i.e., $\tilde{A}(u)^K \cong A(u)$.

4.2. Since $A(u)$ splits, the theory of decomposition morphisms by Geck–Rouquier [12] and Geck–Pfeiffer [11, §7] implies that the decomposition morphism

$$d_A^u : G_0(A(u)) \rightarrow G_0(A(u))$$

exists, where u is the generic point of $\mathrm{Spec}(R)$, i.e., $A(u) = A^{Q(R)}$, where $Q(R)$ is the quotient field of R . Now, by assumption $A(u)$ has an $\mathcal{O}_{\mathfrak{m}}$ -free $\mathcal{O}_{\mathfrak{m}}$ -structure $\tilde{A}(u)$. Since $\mathcal{O}_{\mathfrak{m}}$ is a valuation ring, the decomposition morphism

$$d_{\tilde{A}(u)}^{\mathfrak{m}} : G_0(A(u)) \rightarrow G_0(\tilde{A}(u)(\mathfrak{m}_{\mathfrak{m}}))$$

exists, where $\tilde{A}(u)(\mathfrak{m}_{\mathfrak{m}})$ is the scalar extension of $\tilde{A}(u)$ to the residue field of $\mathfrak{m}_{\mathfrak{m}}$. As decomposition morphisms are strongly positive, we obtain a strongly positive morphism

$$(23) \quad \begin{array}{ccccc} G_0(A(u)) & \xrightarrow{d_A^u} & G_0(A(u)) & \xrightarrow{d_{\tilde{A}(u)}^{\mathfrak{m}}} & G_0(\tilde{A}(u)(\mathfrak{m}_{\mathfrak{m}})) \\ & \searrow & & \nearrow & \\ & d_A^{\mathfrak{m}, u} & & & \end{array}$$

We have omitted the choice of the $\mathcal{O}_{\mathfrak{m}}$ -free $\mathcal{O}_{\mathfrak{m}}$ -structure of $A(u)$ in the notation $d_A^{\mathfrak{m}, u}$ as this will not be important—although the knowledge about the existence of such a structure is of course crucial. We call $d_A^{\mathfrak{m}, u}$ the *decomposition morphism* of A in (\mathfrak{m}, u) but note that this does not have to be a decomposition morphism itself.

4.3. Remark. In [29] it is proven that decomposition morphisms are generically trivial for finite free algebras with split generic fibers over noetherian normal rings. Hence, assuming that R is noetherian and that A has split fibers, the morphism $d_A^{\mathfrak{m}, u}$ is trivial for generic u and for generic \mathfrak{m} , meaning that it induces a bijection between simple modules. Hence, finite field specializations can be used to employ 3.18(b) generically. This already indicates that it makes sense to choose finite field specializations randomly as the probability is quite high to stay in the generic region.

4.4. If (\mathfrak{m}, u) is a finite field specialization of A and V is a finite-dimensional $A(u)$ -module, it will be important to explicitly compute a representative of $d_A^{\mathfrak{m}, u}([V])$. To this end, suppose that the image of u is contained in $\mathcal{O}_{\mathfrak{m}}$ and that we have an $R_{\mathfrak{p}}$ -free $A_{\mathfrak{p}}$ -structure \tilde{V} of V for some $\mathfrak{p} \in \mathrm{Spec}(R)$. Let $\tilde{\mathcal{V}}$ be an $R_{\mathfrak{p}}$ -basis of \tilde{V} and let \mathcal{A} be an R -algebra generating system of A . If we apply the map u to the entries of the matrices describing the action of $a \in \mathcal{A}$ on V in the basis $\tilde{\mathcal{V}}$, we obtain a representative of $d_A^u([V])$. As the image of u is contained in $\mathcal{O}_{\mathfrak{m}}$ by

assumption, the entries of the matrices just obtained are contained in $\mathcal{O}_{\mathfrak{m}}$ and so we can reduce the modulo $\mathfrak{m}_{\mathfrak{m}}$, and this a representative of $d_{A(u)}^{\mathfrak{m}} \circ d_A^u([V]) = d_A^{\mathfrak{m},u}([V])$. In this situation we do not even see the chosen \mathcal{O} -free \mathcal{O} -structure $\tilde{A}(u)$ of $A(u)$.

Although formally a bit complicated, this whole process is actually quite straightforward in explicit situations and is automatically performed by the command `Specialize` in `CHAMP`. That a pair (\mathfrak{m}, u) is indeed a finite field specialization has to be checked manually, however.

Let us now turn to the problem of finding finite field specializations of restricted rational Cherednik algebras.

4.5. Assumption. By $\Gamma := (G, V)$ we denote a finite reflection group over a field K containing a Dedekind domain \mathcal{O} with quotient field K . We assume as usual that all reflections are diagonalizable. Furthermore, we assume that the action of G on V and on V^* has no non-zero fixed points, i.e., the G -modules V and V^* are *essential*. This certainly holds if Γ is irreducible.

4.6. Definition. We say that a Cherednik parameter $c \in \mathfrak{R}_{\Gamma}(K)$ is *\mathcal{O} -integral* if the K -algebra \overline{H}_c has an \mathcal{O} -free \mathcal{O} -structure. We call any such structure an *\mathcal{O} -integral structure*.

It seems that the existence of integral structures of restricted rational Cherednik algebras has never been considered before. After we point out an evident obstruction to the existence of integral structures we will give a sufficient condition for their existence. To this end, note that for any $s \in \text{Ref}_{\Gamma}$ the set

$$\text{Che}_{\Gamma}(s) := \{(y_j, x_i)_s \mid i, j \in [1, n]\} \subseteq K$$

for a K -basis $(y_i)_{i=1}^n$ of V with dual basis $(x_i)_{i=1}^n$ is independent of the chosen basis.

4.7. Definition. We say that $c \in \mathfrak{R}_{\Gamma}(K)$ is *potentially \mathcal{O} -integral* if $c(s)\text{Che}_{\Gamma}(s) \subseteq \mathcal{O}$ for all $s \in \text{Ref}_{\Gamma}$.

4.8. Theorem. If there exists a datum $(\mathbf{y}, \mathcal{A}, \mathcal{B}, \mathcal{G})$ consisting of a basis \mathbf{y} of V with dual basis \mathbf{x} , a basis \mathcal{A} of $K[V]_G$, a basis \mathcal{B} of $K[V^*]_G$, and a generating system \mathcal{G} of G satisfying all of the following properties, then any potentially \mathcal{O} -integral parameter $c \in \mathfrak{R}_{\Gamma}(K)$ is already \mathcal{O} -integral:

- (a) \mathcal{A} contains the images of the elements of \mathbf{x} in $K[V]_G$ and every element of \mathcal{A} is an \mathcal{O} -linear polynomial in these images. The basis \mathcal{B} satisfies the analogous conditions.
- (b) The structure constants of $K[V]_G$ with respect to \mathcal{A} are contained in \mathcal{O} . The structure constants of $K[V^*]_G$ with respect to \mathcal{B} satisfy the analogous conditions.
- (c) For all $g \in \mathcal{G}$ the action of g on V in the basis \mathbf{y} and the action of g on V^* in the basis \mathbf{x} is described by matrices with entries in $\mathcal{O} \subseteq K$.

Proof. Let $\mathbf{x} = (x_i)_{i=1}^n$ and $\mathbf{y} = (y_i)_{i=1}^n$. Let \bar{x}_i and \bar{y}_i denote the images of x_i and y_i in $K[V]_G$ and $K[V^*]_G$, respectively. A K -basis of \overline{H}_c is given by $(abg)_{a \in \mathcal{A}, b \in \mathcal{B}, g \in G}$ and it suffices to show that the structure constants of \overline{H}_c with respect to this basis are contained in \mathcal{O} . Due to (b), products of the form aa' and bb' with $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ are \mathcal{O} -linear combinations of elements of \mathcal{A} and \mathcal{B} , respectively. Let $g \in \mathcal{G}$. Then by (c) we have ${}^g x_i = \sum_{j=1}^n \alpha_j x_j$ with $\alpha_j \in \mathcal{O}$. Since the \bar{x}_i are contained in \mathcal{A} by (a), it follows that ${}^g \bar{x}_i = \sum_{j=1}^n \alpha_j \bar{x}_j$ is the basis expansion of ${}^g \bar{x}_i$ in the basis \mathcal{A} . Hence, the structure constants of the action of G on the elements of $\bar{\mathbf{x}} := (\bar{x}_i)_{i=1}^n$ are contained in \mathcal{O} . If $\lambda \in \mathbb{N}^n$, then

$${}^g \bar{\mathbf{x}}^{\lambda} = {}^g \left(\prod_{i=1}^n \bar{x}_i^{\lambda_i} \right) = \prod_{i=1}^n {}^g \bar{x}_i^{\lambda_i}$$

By what we have just said, the elements ${}^g \bar{x}_i$ are \mathcal{O} -linear combinations of the elements of \mathcal{A} . It now follows from (b) that ${}^g \bar{\mathbf{x}}^{\lambda}$ is an \mathcal{O} -linear combination of the elements of \mathcal{A} . This extends to the action of G on all elements of $K[V]_G$ and therefore the structure constants of

the multiplication of elements of $K[V]_G \subseteq \overline{H}_c$ with group elements are also contained in \mathcal{O} . Analogously, this holds for the action of G on $K[V^*]_G$. The only products of basis elements not already covered are those of the form ba for $b \in \mathcal{B}$ and $a \in \mathcal{A}$. We have

$$\overline{y}_j \overline{x}_i = \overline{x}_i \overline{y}_j + \sum_{s \in \text{Ref}_\Gamma} (y_j, x_i) c(s) s$$

and this is an \mathcal{O} -linear combination of basis elements. By a recursive application of this and the fact that all other basis elements of \mathcal{A} and \mathcal{B} are polynomials in the \overline{x}_i and the \overline{y}_i , respectively, we see that all the products ba are \mathcal{O} -linear combination of basis elements. This shows that \overline{H}_c has an \mathcal{O} -free \mathcal{O} -structure. ■

4.9. Proposition. For any basis \mathbf{y} of V there is a basis \mathcal{A} of $K[V]_G$ and a basis \mathcal{B} of $K[V^*]_G$ satisfying 4.8(a).

Proof. Let $\mathbf{x} = (x_i)_{i=1}^n$. We can then write $K[V] = K[x_1, \dots, x_n]$. Let \mathbf{f} be a system of fundamental invariants of Γ . Note that the degrees of the elements of \mathbf{f} are strictly greater than 1, since if $f \in \mathbf{f}$ would be of degree equal to 1, then it would be an element of V^* fixed by G and thus equal to zero as Γ^* is essential by assumption. Since the Hilbert ideal of Γ is the homogeneous ideal generated by \mathbf{f} , it follows that \mathfrak{h}_Γ does not contain linear polynomials. Now, extend \mathbf{f} to a Gröbner basis $\tilde{\mathbf{f}}$ of the Hilbert ideal \mathfrak{h}_Γ of Γ with respect to the lexicographical order. A monomial basis \mathcal{A} of $K[V]_G$ is then given by the images of the elements

$$\{\mathbf{x}^\alpha \mid \alpha \in \mathbb{N}^n \text{ and } \mathbf{x}^\alpha \text{ is not divisible by some } \text{LT}(f) \text{ for } f \in \tilde{\mathbf{f}}\}$$

in $K[V]_G$ (see 2.5). Now, suppose that the image of x_i in $K[V]_G$ would not be contained in \mathcal{A} . Then by definition there exists $f \in \tilde{\mathbf{f}}$ such that $\text{LT}(f)$ divides x_i . But this means that f is a linear polynomial and we just argued that no linear polynomial is contained in the Hilbert ideal, so this is not possible. We can apply the same arguments to $K[V^*]_G$ and this proves the claim. ■

4.10. Proposition. For all but finitely many maximal ideals \mathfrak{m} of \mathcal{O} any potentially $\mathcal{O}_\mathfrak{m}$ -integral parameter $c \in \mathfrak{R}_\Gamma(K)$ is $\mathcal{O}_\mathfrak{m}$ -integral. We call those \mathfrak{m} for which this is true *good* for the restricted rational Cherednik algebras of Γ .

Proof. Let \mathbf{y} be a basis of V . We know from 4.9 that we can find \mathcal{A} and \mathcal{B} satisfying 4.8(a). Since everything is finite-dimensional, the set S of the structure constants occurring in 4.8(b) and 4.8(c) is finite. Since \mathcal{O} is Dedekind domain, we have $S \subseteq \mathcal{O}_\mathfrak{m}$ for all but finitely many maximal ideals \mathfrak{m} of \mathcal{O} and so 4.8 is satisfied for the bases \mathcal{A} and \mathcal{B} , and the ring $\mathcal{O}_\mathfrak{m}$. ■

4.11. The proof of 4.9 and 4.10 gives us an explicit way to find good maximal ideals of \mathcal{O} . Let us dissect this method as follows:

- (a) Choose an explicit realization $\mathbf{G} \subseteq \text{GL}_n(K)$ of G . This amounts to choosing a basis \mathbf{y} of V . Let \mathbf{x} be the dual basis.
- (b) Compute fundamental invariants \mathbf{f} of \mathbf{G} and \mathbf{f}^* of the dual group \mathbf{G}^* .
- (c) Compute a Gröbner basis of $\mathfrak{h}_\mathbf{G} = \langle \mathbf{f} \rangle$ and of $\mathfrak{h}_{\mathbf{G}^*} = \langle \mathbf{f}^* \rangle$.
- (d) Compute monomial bases \mathcal{A} of the coinvariant algebra $K[\mathbf{x}]/\mathfrak{h}_\mathbf{G}$ and \mathcal{B} of $K[\mathbf{y}]/\mathfrak{h}_{\mathbf{G}^*}$ using the Gröbner bases.
- (e) Compute using the Gröbner bases the structure constants of the coinvariant algebras with respect to the bases \mathcal{A} and \mathcal{B} , respectively.
- (f) Let S be the set of all denominators occurring in these structure constants.
- (g) Choose a generating system \mathcal{G} of G and extend S by the denominators occurring in the corresponding matrices and their inverses.
- (h) Then all \mathfrak{m} not containing any element of S are good.

Precisely this method is performed by the command `BadPrimesForRRCA` in `CHAMP`. In [30, §22] we computed sets of primes which certainly contain all bad primes (for explicit choices of the bases) for the exceptional complex reflection groups G_4 up to G_{28} to ensure correctness of our computations. The primes occurring there are partially surprisingly large and we do not yet have a theoretical explanation for them.

4.12. The primary case we are considering is the following. Let K be a number field with ring of integers \mathcal{O} and let R be the polynomial ring over K with indeterminates $(c_s)_{s \in \mathcal{C}_\Gamma}$. Let $c : \mathcal{C}_\Gamma \rightarrow R$ be the obvious map and let \mathbf{c} be the composition of this map with the embedding into the quotient field of R . Let $\bar{\mathbf{H}} := \bar{\mathbf{H}}_{\mathbf{c}}$ be the *generic restricted rational Cherednik algebra* for Γ . Let $\mathfrak{m} \in \text{Max}(\mathcal{O})$ be a good maximal ideal. Then for any $u \in \text{Che}_\Gamma^{-1} \mathcal{O}^{\mathcal{C}_\Gamma}$ the pair (\mathfrak{m}, u) is a finite field specialization and we have the morphism

$$\begin{array}{ccccc} G_0(\bar{\mathbf{H}}_{\mathbf{c}}) & \xrightarrow{d_{\bar{\mathbf{H}}}^u} & G_0(\bar{\mathbf{H}}_u) & \xrightarrow{d_{\bar{\mathbf{H}}_u}^{\mathfrak{m}}} & G_0(\tilde{\mathbf{H}}_u(\mathfrak{m}_{\mathfrak{m}})) \\ & \searrow & & \nearrow & \\ & & d_{\bar{\mathbf{H}}}^{\mathfrak{m}, u} & & \end{array}$$

where $\tilde{\mathbf{H}}_u$ is some $\mathcal{O}_{\mathfrak{m}}$ -integral structure of $\bar{\mathbf{H}}_u$. As explained in 4.3 the probability of this morphism being trivial in the sense that it induces a bijection between the simple modules is quite high. Thus a random choice of u will bring us in position of employing 3.18. It remains to understand how we can lift back the results from the right to the left in this diagram and this is the topic of the next paragraph. Before we go there, we point out that the same idea works of course if instead of a parameter c yielding the generic point of the whole parameter space \mathfrak{R}_Γ as above we take a parameter yielding the generic point of some closed subscheme of \mathfrak{R}_Γ , e.g., some hyperplane. To have this possibility at hand was one of the reasons why we chose a general commutative K -algebra as base ring everywhere and why we put emphasis on `CHAMP` being able to handle general base rings. In exactly this way—starting with the generic situation and then considering restrictions to hyperplanes—we approach the G_4 -case in §9.

4.13. Remark. What we discussed in this paragraph can of course be generalized to arbitrary finite chains of decomposition morphisms ending in the Grothendieck group of an algebra over a finite field. One only has to make sure in each step that the decomposition morphism exists with the main problem being the existence of integral structures.

§5. Reconstructing submodules from abstract structures

Now that we found a way of transporting the Verma modules to an algebra over a finite field we have to figure out how we can lift back the results obtained there to the initial setting. The idea is the following: if the morphism d induced by a finite field specialization as in (23) satisfies the condition in 3.18(b), then we can think of it as not destroying the structure of modules. Hence, the “abstract structure” of the radical of the image of a local module Δ under this morphism should be the same as the one of Δ itself. From this “abstract structure” we might be able to compute a candidate for the radical of Δ and using the morphism d we can check if this candidate was the correct one. Let us now make precise what we mean by “abstract structure” and how the candidate production works.

5.1. Let V be an n -dimensional vector space over a field K with basis \mathbf{v} and let U be an m -dimensional subspace. For a basis \mathbf{u} of U let $M_{\mathbf{u}}^{\mathbf{v}} \in \text{Mat}_{n \times m}(K)$ be the matrix of the embedding $U \hookrightarrow V$ with respect to the chosen bases. The class $\mathcal{M}_U^{\mathbf{v}}$ of $M_{\mathbf{u}}^{\mathbf{v}}$ in $\text{Mat}_{n \times m}(K) / \text{GL}_m(K)$ consists precisely of the matrices $M_{\mathbf{u}'}^{\mathbf{v}}$ for bases \mathbf{u}' of U . It is an elementary fact that inside $\mathcal{M}_U^{\mathbf{v}}$ there exists precisely one matrix in reduced column echelon form which we denote by $M_U^{\mathbf{v}}$. Hence, once we fixed a basis of V , the subspaces of V are in bijection with $n \times m$ -matrices

in reduced column echelon form. We will now define the notion of the abstract structure of U with respect to \mathbf{v} by using the matrix $M_U^{\mathbf{v}}$.

5.2. Let $M \in \text{Mat}_{n \times m}(K)$. If $\mathcal{E}(M)$ denotes the set of entries of M and if $\theta : \mathcal{E}(M) \rightarrow S$ is a map into a set S , then we denote by $\theta^*(M) \in \text{Mat}_{n \times m}(S)$ the matrix defined by $(\theta^*(M))_{ij} := \theta(M_{ij})$. We denote by $M_{i,\bullet}$ the i -th row of M and by $M_{\bullet,j}$ the j -th column of M . We define $\text{Supp}(M_{i,\bullet}) := \{j \in [1, m] \mid M_{ij} \neq 0\}$. Analogously we define $\text{Supp}(M_{\bullet,j})$ and $\text{Supp}(M)$.

Now, suppose that M is in reduced column echelon form. We define two matrices ${}_cM, {}_fM \in \text{Mat}_{n \times m}(\mathbb{N}_{>0})$ as follows. First, decompose M as $M = {}_cM + {}_fM$, where each column of ${}_cM$ just consists of the leading entry 1 of the corresponding column of M (if there is one) and ${}_fM$ is the matrix $M - {}_cM$. We call ${}_cM$ the *coarse structure* of M . Let \mathcal{E} be the set of entries of ${}_fM$ and for $x \in \mathcal{E}$ let $\mathcal{E}_x := \{(i, j) \in [1, n] \times [1, m] \mid M_{ij} = x\}$. We equip each \mathcal{E}_x with the lexicographical order, which is a total order so that \mathcal{E}_x has a unique minimum, and define an order \leq on \mathcal{E} by $x \leq y$ if and only if $\min \mathcal{E}_x \leq \min \mathcal{E}_y$. This is a total order on the finite set \mathcal{E} so that assigning to each $x \in \mathcal{E}$ its position in \mathcal{E} relative to \leq defines a function $e : \mathcal{E} \rightarrow \mathbb{N}_{>0}$. We now define ${}_fM := e^*({}_fM)$ and call this the *fine structure* of M . We call the pair $\text{Abs}(M) := ({}_cM, {}_fM)$, which we also write as ${}_cM + {}_fM$, the *abstract structure* of M and call $\#\mathcal{E}$ the *complexity* of M . By $\text{Abs}_{n \times m}$ we denote the set of abstract structures of $n \times m$ -matrices in reduced column echelon form.

5.3. Example. Let

$$M := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{{}_cM} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1 & 4 \end{pmatrix}}_{{}_fM} \in \text{Mat}_{3 \times 2}(\mathbb{Q}).$$

Then

$$\text{Abs}(M) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{{}_cM} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}}_{{}_fM} \in \text{Mat}_{3 \times 2}(\mathbb{N}_{>0}).$$

In this example we have $\mathcal{E} = \{2, 1, 4\}$ and $e : \mathcal{E} \rightarrow [1, 3]$ is defined by $e(2) = 1$, $e(1) = 2$, $e(4) = 3$. The complexity of M is equal to 3.

5.4. Definition. If V is a finite-dimensional vector space over a field K with basis \mathbf{v} , then the *abstract structure* $\text{Abs}_U^{\mathbf{v}}$ with respect to \mathbf{v} of a subspace U of V is the abstract structure of the matrix $M_U^{\mathbf{v}}$.

5.5. Definition. If an abstract structure $M := ({}_cM, {}_fM) \in \text{Abs}_{n \times m}$ with $m \leq n$ is given, then for any map $\theta : \mathcal{E}({}_fM) \rightarrow K^\times$ with $\theta(i) \neq \theta(j)$ for $i \neq j$ we get a matrix ${}_cM + \theta^*({}_fM) \in \text{Mat}_{n \times m}(K)$ in reduced column echelon form describing a unique subspace $U_{M,\theta}^{\mathbf{v}}$ of V with respect to the basis \mathbf{v} . We call this subspace the *concretization* of M with respect to θ and \mathbf{v} .

Note that an abstract structure itself is *independent* of a base field—this is precisely the point of abstract structures. We can now formulate the primary aim of this paragraph and we do this in a graded setting as the efficiency of CHAMP also relies on the fact that we make use of gradings throughout.

5.6. Let A be a finite-dimensional \mathbb{Z} -graded algebra over a field K , let V be a \mathbb{Z} -graded n -dimensional A -module, and let $\mathbf{v} := (v_i)_{i=1}^n$ be a homogeneous basis of V . The question this whole paragraph is about is:

Given an abstract structure $M := ({}_cM, {}_fM) \in \text{Abs}_{n \times m}$ with $m \leq n$, is there a graded submodule U of V with $\text{Abs}_U^V = M$? In other words, is there a map $\theta : \mathcal{E}({}_fM) \rightarrow K^\times$ with $\theta(i) \neq \theta(j)$ for $i \neq j$ such that the concretization $U_{M,\theta}^V$ is a graded submodule of V ?

To analyze this question we choose a set $\mathbf{a} := (a_k)_{k=1}^r$ of homogeneous K -algebra generators of A and denote for each $k \in [1, r]$ by $X^{(k)} \in \text{Mat}_n(K)$ the matrix describing the action of a_k on V in the basis \mathbf{v} , i.e.,

$$(24) \quad a_k v_i = \sum_{l=1}^n X_{li}^{(k)} v_l = \sum_{l \in D_{ki}^r} X_{li}^{(k)} v_l$$

for all $j \in [1, n]$, where

$$D_{ki}^r := \{l \in [1, n] \mid \deg(a_k) + \deg(v_i) = \deg(v_l)\}.$$

5.7. Theorem. The answer to the above question is positive if and only if the following conditions are satisfied:

- (a) For each $j \in [1, m]$ the degree of v_i is constant for all $i \in \text{Supp}(M_{\bullet,j})$. We define $d_M^c(j)$ to be this degree.
- (b) There exist pairwise different $\theta_1, \dots, \theta_s \in K^\times$, where s is the complexity of M , and a family

$$(Y_l^{(k,j)})_{\substack{k \in [1, r], j \in [1, m] \\ l \in D_{kj}^c}} \subseteq K,$$

where

$$D_{kj}^c := (d_M^c)^{-1}(\deg(a_k) + d_M^c(j))$$

such that the equations

$$(25) \quad E_{i,j,k}^1 : \sum_{l \in I_{ijk}} {}_cM_{lj} X_{li}^{(k)} + \sum_{l \in I_{ijk}} \theta_{{}_fM_{lj}} X_{il}^{(k)} = 0$$

hold for all $j \in [1, m]$, $k \in [1, r]$, $i \in \text{Supp}(M_{\bullet,j})$, and such that the equations

$$(26) \quad E_{ijk}^2 : \sum_{l \in I_{ijk}} {}_cM_{lj} X_{il}^{(k)} + \sum_{l \in I_{ijk}} \theta_{{}_fM_{lj}} X_{il}^{(k)} = \sum_{l \in D_{kj}^c} Y_l^{(k,j)} {}_cM_{il} + \sum_{l \in D_{kj}^c} Y_l^{(k,j)} \theta_{{}_fM_{il}}$$

hold for all $j \in [1, m]$, $k \in [1, r]$, and $i \in [1, n] \setminus \text{Supp}(M_{\bullet,j})$, where

$$I_{ijk} := \{l \in \text{Supp}(M_{\bullet,j}) \mid i \in D_{kl}^r\}.$$

Proof. Suppose that the conditions are satisfied. Let $\theta : [1, s] \rightarrow K^\times$ be the map with $\theta(i) := \theta_i$. Then the concretization $U := U_{M,\theta}^V$ defines a unique subspace of V . Let $N_{\bullet,j} := {}_cM_{\bullet,j} + (\theta^* {}_fM)_{\bullet,j}$ be the “specialization” of the j -th column of M in θ . Define

$$(27) \quad u_j := \sum_{i=1}^n N_{i,j} v_i = \sum_{i \in \text{Supp}(M_{\bullet,j})} N_{i,j} v_i = \sum_{i \in \text{Supp}(M_{\bullet,j})} {}_cM_{ij} v_i + \theta_{{}_fM_{ij}} v_i.$$

Then $(u_j)_{j=1}^m$ is a basis of U and because of (a) this is a graded subspace. It remains to show that U is A -invariant. This holds if and only if $a_k U \subseteq U$ for all $k \in [1, r]$, and this in turn holds if and only if $a_k u_j \in U$ for all k , so $a_k u_j \in \langle u_1, \dots, u_m \rangle_K$. As u_j is homogeneous of degree $\deg(a_k) + d_M^c(j)$, this is equivalent to $a_k u_j \in \langle u_l \mid l \in D_{kj}^c \rangle$. This is equivalent to the existence of elements $Y_l^{(k,j)} \in K$ such that

$$(28) \quad a_k u_j = \sum_{l \in D_{kj}^c} Y_l^{(k,j)} u_l.$$

Combining equations (24), (27), and (28) implies that this is equivalent to the following equality for each $j \in [1, m]$ and $k \in [1, r]$:

$$\begin{aligned}
a_k \left(\sum_{i \in \text{Supp}(M_{\bullet,j})} c_{M_{ij}} v_i + \theta_{f_{M_{ij}}} v_i \right) &= \sum_{l \in D_{kj}^c} Y_l^{(k,j)} \left(\sum_{i \in \text{Supp}(M_{\bullet,j})} c_{M_{il}} v_i + \theta_{f_{M_{il}}} v_i \right) \\
\Leftrightarrow \sum_{i \in \text{Supp}(M_{\bullet,j})} \sum_{l \in D_{ki}^r} c_{M_{ij}} X_{li}^{(k)} v_l + \sum_{i \in \text{Supp}(M_{\bullet,j})} \sum_{l \in D_{ki}^r} \theta_{f_{M_{ij}}} X_{li}^{(k)} v_l \\
&= \sum_{i \in \text{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^c} Y_l^{(k,j)} c_{M_{il}} v_i + \sum_{i \in \text{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^c} Y_l^{(k,j)} \theta_{f_{M_{il}}} v_i \\
\Leftrightarrow \sum_{i=1}^n \sum_{l \in I_{ijk}} c_{M_{lj}} X_{il}^{(k)} v_i + \sum_{i=1}^n \sum_{l \in I_{ijk}} \theta_{f_{M_{lj}}} X_{il}^{(k)} v_i \\
&= \sum_{i \in \text{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^c} Y_l^{(k,j)} c_{M_{il}} v_i + \sum_{i \in \text{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^c} Y_l^{(k,j)} \theta_{f_{M_{il}}} v_i .
\end{aligned}$$

As \mathbf{v} is a basis of V , each of these equations holds if and only if the coefficients of v_i for each $i \in [1, n]$ are the same. If $i \notin \text{Supp}(M_{\bullet,j})$ the coefficient equation is

$$\sum_{l \in I_{ijk}} c_{M_{lj}} X_{il}^{(k)} + \sum_{l \in I_{ijk}} \theta_{f_{M_{lj}}} X_{il}^{(k)} = 0 .$$

If $i \in [1, n] \setminus \text{Supp}(M_{\bullet,j})$ the coefficient equation is

$$\sum_{l \in I_{ijk}} c_{M_{lj}} X_{il}^{(k)} + \sum_{l \in I_{ijk}} \theta_{f_{M_{lj}}} X_{il}^{(k)} = \sum_{l \in D_{kj}^c} Y_l^{(k,j)} c_{M_{il}} + \sum_{l \in D_{kj}^c} Y_l^{(k,j)} \theta_{f_{M_{il}}} .$$

These are the two asserted types of equations. It is evident from the discussion that these equations are also necessary for the existence of a graded submodule. \blacksquare

5.8. Let $E_{M,\mathbf{v}}^1 := (E_{i,j,k}^1)$ be the system of equations defined by (25), let $E_{M,\mathbf{v}}^2 := (E_{i,j,k}^2)$ be the system of equations defined by (26), and let $E_{M,\mathbf{v}}$ be the whole system. For finding a graded submodule of V with abstract structure M we have to solve the system $E_{M,\mathbf{v}}$ for the θ -variables $\theta_1, \dots, \theta_s$ and the auxiliary variables $Y_l^{(k,j)}$. If there is a unique submodule with this abstract structure—for example if M is the abstract structure of the unique maximal submodule when V is local—this system will have a unique solution we are searching for.

5.9. While $E_{M,\mathbf{v}}^1$ is an inhomogeneous linear system for the θ -variables, the system $E_{M,\mathbf{v}}^2$ is quadratic because of the products $Y_l^{(k,j)} \theta_{f_{M_{il}}}$ occurring in the equations. Hence, it will be very difficult in general to solve this system. But we can still try to consecutively solve *linear parts* of this system. Namely, we can start solving $E_{M,\mathbf{v}}^1$, which is easy as it is a linear system. The point is now that this system might already pin down one of the θ -variables. When plugging in the determined θ -variables into the system $E_{M,\mathbf{v}}^2$ we might get further *linear* equations just involving the auxiliary variables. If we can determine some of the auxiliary variables, then plugging them into $E_{M,\mathbf{v}}^2$ might yield new *linear* equations for the θ -variables which might pin down further θ -variables etc. This means we consecutively solve the “specialized systems” $L_{M,\mathbf{v}}(\theta', Y')$ given by the linear equations of the system $E_{M,\mathbf{v}}$ when plugging in a family θ' of θ -variables and a family Y' of auxiliary variables. If this process leads to a (unique) solution of $E_{M,\mathbf{v}}$ we say that this system is (uniquely) *linearly solvable*. It might happen, however, that at some stage we cannot determine any new variables—then the system is not linearly solvable.

5.10. As we will work with modules of dimension up to 3,000 we need a very efficient strategy for determining the θ -variables by linear equations of $E_{M,\mathbf{v}}$ (if this is possible at all). To this end, we define for any $q \in [1, s]$ a subsystem of $L_{M,\mathbf{v}}^q(\theta', Y')$ just consisting of the linear equations of $E_{M,\mathbf{v}}$ involving θ_q and all *dependent variables*. To make this precise, denote for

a subsystem E of $E_{M,v}$ by $\Theta(E)$ the set of non-determined θ -variables occurring in these equations. For $q \in [1, s]$ let $\tilde{L}_{M,v}^q(\theta', Y')$ just consist of the equations of $L_{M,v}(\theta', Y')$ involving the variable θ_q , i.e.,

$$\tilde{L}_{M,v}^q(\theta', Y') := \{L \in L_{M,v}(\theta', Y') \mid \theta_q \in \Theta(L)\}.$$

Now, define $L_{M,v}^q(\theta', Y')$ inductively as follows. First, $L_{M,v}^q(\theta', Y') := \tilde{L}_{M,v}^q(\theta', Y')$. For each $\theta_p \in \Theta(L_{M,v}^q(\theta', Y'))$ we add to $L_{M,v}^q(\theta', Y')$ the equations of $\tilde{L}_{M,v}^p(\theta', Y')$. We repeat this process until $L_{M,v}^q(\theta', Y')$ stabilizes.

We will split the system $E_{M,v}$ once more by defining $L_{M,v}^{q,g}(\theta', Y')$ for a subset $g \subseteq [1, r]$ as the subsystem of $L_{M,v}^q(\theta', Y')$ just involving equations E_{ijk}^1 or E_{ijk}^2 with $k \in g$. The idea behind this is that we do not want to consider all algebra generators at once—perhaps a few algebra generators will be sufficient to determine all θ -variables and this means we have to consider fewer equations. This idea turned out to be very efficient in experiments (see §8).

5.11. Our idea of solving $E_{M,v}$ is now summarized in algorithm 2. This algorithm—which

Algorithm 2: Finding submodules with prescribed abstract structure (MODFINDER)

Data: Data as in 5.6 and 5.7 satisfying 5.7(a), and a subset $g \subseteq [1, r]$.

Result: Decides if the system $E_{M,v}$ is uniquely linearly solvable. If so, returns a graded submodule U of V with $\text{Abs}_U^v = M$.

```

1   $\theta' := \emptyset; Y := \emptyset;$ 
2  while  $\#\theta' \neq s$  do
3      progress := false;
4      for  $q \in \Theta(E_{M,v}(\theta', Y'))$  do
5          if  $L_{M,v}^{q,g}(\theta', Y')$  is not consistent then
6              return There is no graded submodule with abstract structure  $M$ ;
7          end
8          Let  $\theta''$  and  $Y''$  be the  $\theta$ -variables and auxiliary variables, respectively, determined
          by  $L_{M,v}^{q,g}(\theta', Y')$ ;
9          if  $\theta''$  or  $Y''$  contains a variable not in  $\theta'$  or  $Y'$ , respectively, then
10              $\theta' := \theta' \cup \theta''; Y' := Y' \cup Y'';$ 
11             progress := true;
12         end
13     end
14     if progress = false then
15         if  $g = [1, r]$  then
16             return  $E_{M,v}$  is not uniquely linearly solvable;
17         else
18             Repeat the above algorithm with  $g = [1, r]$ ;
19         end
20     end
21 end
22 Check if  $U_{M,\theta}^v$  is indeed a submodule of  $V$ ;
23 if this is true then
24     return  $U_{M,\theta}^v$ ;
25 else
26     return  $E_{M,v}$  is not uniquely linearly solvable;
27 end

```

we call the **MODFINDER** algorithm—has been implemented in this way (and with several additional ideas we cannot discuss here) in CHAMP in the subpackage `ModFinder`. In line 22 we have to check whether the concretization $U_{M,\theta}^v$ is indeed a submodule as we are just solving subsystems of $E_{M,v}$ and just verify necessary conditions up to this point. This can efficiently be checked using the *graded spinning algorithm*—a graded adaption of the standard spinning algorithm explained for example in [21, §1.3]. All this is provided by the new type `ModGr` for graded modules we have implemented in CHAMP.

5.12. Remark. Obviously, there is no reason why we can solve $E_{M,v}$ just by consecutively solving specialized linear subsystems. For the radicals of Verma modules for restricted rational Cherednik algebras, however, this surprisingly turned out to be almost always the case and our algorithm was amazingly efficient—we cannot yet give theoretical arguments in favor of this.

5.13. Remark. In experiments we observed that the choice of \mathbf{g} and the order in which we try to determine θ -variables (line 4 in algorithm 2) can have a serious impact on the runtime of the algorithm (see §8). We do not know yet how to determine an optimal choice of \mathbf{g} and on the sequence of θ -variables to solve for. The interaction between the subsystems $L_{M,v}^{q,\mathbf{g}}(\theta', Y')$ seems to be very hard to understand. In CHAMP we have implemented a selection process for the systems which performs quite well in experiments.

§6. A Las Vegas algorithm for computing the head and the constituents of local modules

6.1. With the theory about finite field specializations and the `ModFinder` algorithm we can now render our idea explained abstractly in 3.20 to an algorithm. Remember that we are considering a finite-dimensional algebra A over a field and a family $(V_\lambda)_{\lambda \in \Lambda}$ of finite-dimensional local A -modules with heads $(S_\lambda)_{\lambda \in \Lambda}$ such that this family is constituent-closed, meaning that every constituent of a member V_λ of this family is the head S_μ of some V_μ . Algorithm 3 attempts to compute the simple modules S_λ and the multiplicities of S_μ in V_λ .

6.2. Remark. We see that there are three branches in our algorithm whose result will be that the algorithm is not successful. On the other hand, if the algorithm is successful, it follows from our discussion, that the result returned is the correct result. This means that our algorithm is a so-called Las Vegas algorithm, like the `MEATAXE` itself. Because of this it is not easy to provide a complexity analysis of our approach. Note that whenever the algorithm is unsuccessful, it makes sense to run it again with a different finite field specialization—perhaps the one chosen before destroyed some structure.

6.3. Let us now discuss how we apply our algorithms so far to approach Gordon’s questions 3.7 in case of generic restricted rational Cherednik algebras (see 4.12) for irreducible complex reflection groups. First, we choose a realization Γ of the reflection group over a number field K with ring of integers \mathcal{O} (this is always possible). Then we compute which maximal ideals of \mathcal{O} are certainly good using 4.11. Next, we compute the generic Euler families Eu_c (see 3.12). For each Euler family Λ the Verma modules $(\Delta_c(\lambda))_{\lambda \in \Lambda}$ form a constituent closed family of local modules to which we apply our algorithm.

The random finite field specialization (line 1 of the algorithm) is chosen as $d_{\frac{m,u}{H}}^{\mathbf{m},u}$ by randomly choosing a good maximal ideal \mathbf{m} and a point $u \in \text{Che}_\Gamma^{-1} \mathcal{O}^{\text{gr}}$ as explained in 4.12. All this is automatically performed in CHAMP by the commands `HeadOfLocalModule` and `HeadsOfLocalModules` contained in the subpackage `RadicalLift`. This command is in general applicable to any constituent closed family of local modules over an algebra over a rational function field over a number field. Note, however, that one has to ensure by theory that the chosen data (\mathbf{m}, u) is indeed a finite field specialization.

Algorithm 3: Computing heads and decomposition matrices

Data: Data as in 6.1.
Result: If successful, returns the simple modules S_λ and the multiplicity $m_{\lambda,\mu}$ of S_μ in V_λ .

```

1 Randomly choose a strongly positive morphism  $d : G_0(A) \rightarrow G_0(B)$  with  $B$  a
  finite-dimensional algebra over a finite field;
2 for  $\lambda \in \Lambda$  do
3   Compute a representative  $\overline{V}_\lambda$  of  $d([V])$ ;
4   Compute using the MEATAXE the radical  $\overline{J}_\lambda$  of  $\overline{V}_\lambda$ ;
5   Determine the abstract structure  $\overline{J}_\lambda^{\text{abs}}$  of  $J$  in  $\overline{V}_\lambda$ ;
6   Using algorithm 2 try to find a submodule  $J_\lambda$  of  $V_\lambda$  with abstract structure  $\overline{J}_\lambda^{\text{abs}}$ ;
7   if  $J_\lambda$  could not be determined then
8     return No success;
9   else
10     $Q_\lambda := V_\lambda / J_\lambda$ ;
11    Compute a representative  $\overline{Q}_\lambda$  of  $d([Q_\lambda])$ ;
12    Check using the MEATAXE if  $\overline{Q}_\lambda$  is irreducible;
13    if this is not true then
14      return No success;
15    end
16  end
17 end
18 for  $\lambda \in \Lambda$  do
19   Compute using the MEATAXE the constituents  $(\overline{U}_{\lambda,\theta})_{\theta \in \Theta_\lambda}$  and their multiplicities
     $m_{\lambda,\theta}$  of  $\overline{V}_\lambda$ ;
20   Find using the MEATAXE an injection  $\iota_\lambda : \Theta_\lambda \hookrightarrow \Lambda$  such that  $\overline{U}_{\lambda,\theta} \cong \overline{Q}_\mu$  for  $\mu \in \Lambda$ 
    and  $\theta \in \Theta_\lambda$  if and only if  $\mu = \iota_\lambda(\theta)$ ;
21   if no such injection exists then
22     return No success;
23   end
24    $m_{\lambda,\iota_\lambda(\theta)} := m_{\lambda,\theta}$  for all  $\theta \in \Theta_\lambda$  and  $m_{\lambda,\mu} := 0$  for all  $\mu \notin \text{Im } \iota_\lambda$ ;
25 end
26 return  $(Q_\lambda)_{\lambda \in \Lambda}, (m_{\lambda,\mu})_{\lambda,\mu \in \Lambda}$ ;

```

If successful, our algorithm computes the generic Verma families (see 3.13) which we can compare to the Euler families. If these coincide, we also know the Calogero–Moser families. As Bonnafé–Rouquier have proven that the Verma families are actually already equal to the Calogero–Moser families (see 3.13), we actually do not have to hope that they are equal to the Euler families (this is not always the case).

Note that in case of success we have also explicitly computed the simple modules so that we know their dimension, their Poincaré series, and using character theory we can also compute their structure as graded G -modules. Hence, we can answer all of Gordon’s questions once our algorithm was successful.

The same idea is of course applicable if we do not start with the generic algebra $\overline{\mathbf{H}}$ but with its restriction to a hyperplane, say. This is exactly what we will do in case of G_4 in §9.

6.4. Remark. If we work with a generic restricted rational Cherednik algebra $\overline{\mathbf{H}}$ for a reflection group Γ over a *finite* field K which splits over K , the choice of the morphism d in line 1 of the algorithm is actually simpler. As restricted rational Cherednik algebras split, we have a

decomposition morphism $d_{\overline{H}}^{\mathfrak{p}} : G_0(\overline{H}(0)) \rightarrow G_0(\overline{H}(\mathfrak{p}))$ for any prime ideal \mathfrak{p} of the base ring of \overline{H} and we can choose for \mathfrak{p} any K -point of \mathfrak{R}_{Γ} . This approach is also covered by CHAMP.

§7. CHAMP

Now, we pass to the experimental part of this article. Everything we discussed so far has been implemented in CHAMP. The source code and documentation of CHAMP is freely available at <http://thielul.github.io/CHAMP/>. All parts are licensed under the GPL. Due to some operating system functions used in CHAMP, it will not work on Windows systems, just on Linux and Mac OS X systems. Moreover, a MAGMA version of at least 2.19 (released in December 2012) is necessary as we make use of user-defined types which did not exist in earlier versions.¹

Once the downloaded package is unpacked one has to configure CHAMP by running

```
$ ./configure
```

in a terminal and inside the directory of CHAMP. This sets several variables to the absolute path of CHAMP and is necessary for working with it. CHAMP is now started by running:

```
$ ./champ
Loading file "/CHAMP/CHAMP.m"

CHAMP (CHerednik Algebra Magma Package)
Version v1.2-8-gcd9f94e
Copyright (C) 2013, 2014 Ulrich Thiel
Licensed under GNU GPLv3, see COPYING.
thiel@mathematik.uni-stuttgart.de
```

>

Before we give a rough description of the capabilities of CHAMP, we point out the following important aspect:

All actions in MAGMA are *right* actions. This means whenever we start with a reflection group acting from the left and we consider left modules over rational Cherednik algebras, we have to transpose all matrices in MAGMA. Moreover, the rational Cherednik algebra implemented in CHAMP is the *opposite* algebra of the one we are describing here theoretically. Hence, we have to reverse all products when passing between theory and CHAMP.

This reversion process between theory and CHAMP might be confusing at first but we found it much more confusing when artificially working with left actions in MAGMA.

7.1. As one aim of CHAMP was to verify Martino's conjecture we had to make sure that we use the same labelings of irreducible characters of complex reflection groups as the one used by Chlouveraki [6] for the computation of Rouquier families. This is why we imported all relevant data from CHEVIE (see [CHEVIE]) and implemented basic data base support in CHAMP to deal with this data. This is illustrated by the following example:

```
> G:=ExceptionalComplexReflectionGroup(4);
> CharacterTable(~G);
> G\CharacterNames;
[ \phi_{1,0}, \phi_{1,4}, \phi_{1,8}, \phi_{2,5}, \phi_{2,3}, \phi_{2,1},
  \phi_{3,2} ]
```

¹There is a recent agreement between the MAGMA group and the Simons foundation which will make MAGMA much more widely available within the US so that at least in the US it should not be a problem obtaining a recent MAGMA version (see <http://magma.maths.usyd.edu.au/magma/simons/>).

In this example we loaded the exceptional complex reflection group G_4 . The realization is the same as in CHEVIE, but note that all matrices are transposed. Then we attached the character table to this group. When doing this the names of the characters used in CHEVIE are automatically loaded and stored in the attribute `CharacterNames` of the group. We see in this example that one philosophy of CHAMP is to work with procedures taking references to objects as input and store their result in the corresponding attribute of the objects. The reason for this is that we want to have easy access on all data already computed and to handle the large amount of data necessary to work with rational Cherednik algebras. The absolutely irreducible characteristic zero representations are now attached using the procedure `Representations(~G, 0)` and can be accessed via `G'Representations[0]`. Again we use the exact same realizations of these representations as in CHEVIE. Absolutely irreducible representations in characteristic p can be attached by calling the above command with p instead of 0.

7.2. Next to the characters and representations, the reflections are important. A structured collection of the reflections is attached by the command `ReflectionLibrary` which gathers all the reflections of a reflection group Γ in a nested list of the form

$$\left(\left((s)_{H_s=H} \right)_{H \in \Omega} \right)_{\Omega \in \mathcal{A}_\Gamma}.$$

Hence, for each orbit Ω of reflection hyperplanes of Γ we have for each $H \in \Omega$ a list consisting of the reflections with hyperplane H . This allows us to label a reflection of Γ by a triple (i, j, k) , where i refers to the i -th reflection hyperplane orbit, j refers to the j -th hyperplane in the orbit labeled by i , and k refers to the k -th reflection with hyperplane j . This is precisely the triple we get when passing a reflection to the function `ReflectionID`. From the reflection library we automatically store representatives of the conjugacy classes of reflections in the attribute `ReflectionClasses`.

7.3. A generic Cherednik parameter can be obtained as follows:

```
> G:=ExceptionalComplexReflectionGroup(4);
> c:=CherednikParameter(G : Type:="GGOR"); c;
> c;
Mapping from: { 1 .. 2 } to Multivariate rational function field of rank 2
over Cyclotomic Field of order 3 and degree 2
<1, (-zeta_3 + 1)*k_{1,1} + (2*zeta_3 + 1)*k_{1,2}>
<2, (zeta_3 + 2)*k_{1,1} + (-2*zeta_3 - 1)*k_{1,2}>
```

This will be a map $c : [1, N] \rightarrow L$, where N is the number of conjugacy classes of reflections and L is the appropriate rational function field (the residue field in the generic point of \mathfrak{A}_Γ). The numbers 1 to N of the domain of c refer to the numbers in `ReflectionClasses`. So, if s is a reflection of Γ and i is its reflection class number, then $c(i) = c(s)$.

The command `CherednikParameter` has the additional option `Type` which allows specification of different types of parameters. In the above, we selected the `GGOR` type (see 3.9). We can instead also pass `EG` as type which are the parameters used in 2.2 or we can pass `BR` which are the parameters used in [4]. There is a further option `Rational` which, when set to false, returns the parameter with values in the polynomial ring instead of the rational function field. Instead of using generic parameters, the user can define any map $c : [1, N] \rightarrow L$ as above which can be used for a Cherednik parameter.

7.4. Rational Cherednik algebras can be created as follows:

```
> G:=ExceptionalComplexReflectionGroup(4);
> c:=CherednikParameter(G : Type:="EG");
> H:=RationalCherednikAlgebra(G, <1, c>); H;
```

```

Rational Cherednik algebra
Generators:
    y2, y1, g2, g1, x2, x1
Generator degrees:
    -1, -1, 0, 0, 1, 1
Base ring:
    Multivariate rational function field of rank 2 over Cyclotomic Field of
    order 3 and degree 2
    Variables: c1, c2
Number of elementary rewrite rules:
    14
Group:
    MatrixGroup(2, Cyclotomic Field of order 3 and degree 2) of
    order 2^3 * 3
    Generators:
    [      1      0]
    [      0 zeta_3]

    [1/3*(2*zeta_3 + 1) 1/3*(2*zeta_3 - 2)]
    [ 1/3*(zeta_3 - 1)  1/3*(zeta_3 + 2)]
t-parameter:
    1
c-parameter:
    Mapping from: { 1 .. 2 } to Multivariate rational function field of
    rank 2 over Cyclotomic Field of order 3 and degree 2
    <1, c1>
    <2, c2>
> H.6*H.2;
2/3*c2*g1*g2^2*g1^2 + 2/3*c2*g1^2*g2^2*g1 + 2/3*c1*g1*g2*g1^2 +
2/3*c1*g1^2*g2*g1 + y1*x1 + 2/3*c2*g2^2 + 2/3*c1*g2 + 1

```

In the above example we created the opposite rational Cherednik algebra $H^{\text{op}} := H_{1,\mathbf{c}}^{\text{op}}$ for G_4 and the rational point \mathbf{c} of \mathfrak{R}_Γ . The generators of H can be accessed via `H.i`, where i lies between $2d + e$, where d is the dimension of Γ and e is the number of generators of Γ . We see in the above output that the generators are ordered as $y_2, y_1, g_2, g_1, x_2, x_1$ and in the example we computed the product $x_1 y_1$. Now, keep in mind that H as created is the *opposite* algebra to what we treated theoretically before. This is why $x_1 y_1$ is *not* in PBW form—it is actually the product $y_1 x_1$ and this has to be rewritten.

7.5. In CHAMP we implemented very generally a support for algebras with monomial rewrite systems in the subpackage `AlgRew`. Rational Cherednik algebras are just a special instantiation of this type. In principle we can use this to do computations in symplectic reflection algebras and more generally in Drinfeld–Hecke algebras.

7.6. Example. Let us see how CHAMP handles a very elaborate example from Bonnafé–Rouquier [4, §19]. The Weyl group of type B_2 can be realized as the matrix group Γ in $\text{GL}_2(\mathbb{Q})$ generated by the reflections

$$s := g_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t := g_2 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let y_1, y_2 be the standard basis of $V := \mathbb{Q}^2$ and let x_1, x_2 be the dual basis. Let $\{A, B\}$ be algebraically independent over \mathbb{Q} and define $\mathbf{c}_s := -2A$, $\mathbf{c}_t := -2B$. As s and t are representatives of the conjugacy classes of reflections of Γ , this yields a map $\mathbf{c} : \mathcal{C}_\Gamma \rightarrow \mathbb{Q}(A, B)$ giving the generic point of \mathfrak{R}_Γ . Now, define the following elements of $H_{0,\mathbf{c}}$:

$$\sigma := y_1^2 + y_2^2, \quad \pi := y_1^2 y_2^2, \quad \Sigma := x_1^2 + x_2^2, \quad \Pi := x_1^2 x_2^2.$$

In [4, 19.4.5] it is now proven that the Euler element $\text{eu}_c \in H_{0,c}$ is a zero of the polynomial

$$\begin{aligned} & t^8 - 2(\sigma\Sigma + 4A^2 + 4B^2)t^6 \\ & + (\sigma^2\Sigma^2 + 2(\sigma^2\Pi + \Sigma^2\Pi - 8\pi\Pi) + 8(A^2 + B^2)\sigma\Sigma + 16(A^2 - B^2)^2)t^4 \\ & - 2((\sigma\Sigma + 4A^2 - 4B^2)(\sigma^2\Pi + \Sigma^2\Pi) - 8\sigma\Sigma\pi\Pi + 2B^2\sigma^2\Sigma^2)t^2 + (\sigma^2\Pi - \Sigma^2\pi)^2. \end{aligned}$$

This fact was one essential part in determining the Calogero–Moser cells and to prove the Calogero–Moser cell conjecture for B_2 . In [4] this is proven by an argument based on the undeformed situation in $H_{0,0}$. As the computation is quite elaborate and one does not want to write down all its details, let us see if we can verify this fact with CHAMP:

```
> G:=CHAMP_GetFromDB("GrpMat/B2_BR","GrpMat"); //loads B2 as above
> C:=CherednikParameter(G:Type:="BR");
> H:=RationalCherednikAlgebra(G,C);
> eu:=EulerElement(H); eu;
-C1*g2*g1*g2 - C2*g1*g2*g1 + y2*x2 + y1*x1 - C2*g2 - C1*g1
> A:=C(1)*(-1/2); B:=C(2)*(-1/2);
> y2:=H.1;y1:=H.2;g2:=H.3;g1:=H.4;x2:=H.5;x1:=H.6;
> sigma:=y1^2+y2^2; pi:=y2^2*y1^2; Sigma:=x1^2+x2^2; Pi:=x2^2*x1^2;
> time eu^8 - 2*eu^6*(Sigma*sigma + 4*A^2 + 4*B^2) + eu^4*(Sigma^2*sigma^2 +
2*(Pi*sigma^2 + pi*Sigma^2 - 8*Pi*pi) + 8*Sigma*sigma*(A^2+B^2) +
16*(A^2-B^2)^2) - 2*eu^2*( (Pi*sigma^2 + pi*Sigma^2)*(Sigma*sigma +
4*A^2 - 4*B^2) - 8*Pi*pi*Sigma*sigma + Sigma^2*sigma^2*B^2*2) +
(Pi*sigma^2 - pi*Sigma^2)^2;
0
Time: 121.070
```

Hence, we could indeed verify within only 121 seconds that the Euler element is a zero of the polynomial above. In this calculation 2,694,488 elementary rewrite rules were applied. Note that we reversed all products as CHAMP works in the opposite algebra.

7.7. Let us now see how we can compute in CHAMP with Verma modules for restricted rational Cherednik algebras:

```
> G:=ExceptionalComplexReflectionGroup(4); Representations(~G,0);
> c:=CherednikParameter(G:Rational:=false); c;
Mapping from: { 1 .. 2 } to Polynomial ring of rank 2 over Cyclotomic Field
of order 3 and degree 2
<1, (-zeta_3 + 1)*k_{1,1} + (2*zeta_3 + 1)*k_{1,2}>
<2, (zeta_3 + 2)*k_{1,1} + (-2*zeta_3 - 1)*k_{1,2}>
> R:=Codomain(c); R;
Polynomial ring of rank 2 over Cyclotomic Field of order 3 and degree 2
Order: Lexicographical
Variables: k_{1,1}, k_{1,2}
> cH:=SpecializeCherednikParameterInHyperplane(c, R.1-R.2); c;
Mapping from: { 1 .. 2 } to Multivariate rational function field of rank 1
over Cyclotomic Field of order 3 and degree 2
<1, (zeta_3 + 2)*k_{1,2}>
<2, (-zeta_3 + 1)*k_{1,2}>
> V:=[* VermaModule(G,cH,G'Representations[0][i]) : i in [1..7] *];
> V[7];
Graded module of dimension 72 over an algebra with generator degrees
[ -1, -1, 0, 0, 1, 1 ] over Multivariate rational function field of rank 1
over Cyclotomic Field of order 3 and degree 2.
> res,L,D:=HeadsOfLocalModules(V:pExclude:={2,3,7},gens:={{1,2,3}});
> D;
[1 0 0 0 0 0 0]
```

```

[0 1 1 2 0 0 0]
[0 1 1 2 0 0 0]
[0 2 2 4 0 0 0]
[0 0 0 0 2 2 0]
[0 0 0 0 2 2 0]
[0 0 0 0 0 0 3]
> IsModuleForRRCA(G, cH, L[7]);
Everything OK.
true
> GradedGModuleStructureOfRRCAModule(G, L[2]);
[ <0, 2, 1>, <1, 6, 1>, <2, 7, 1>, <3, 6, 1>, <4, 1, 1> ]

```

In this example we are considering the group G_4 . At the beginning we created the (non-rational) generic Cherednik parameter of GGOR type. We specialized this parameter in the hyperplane H defined by $k_{1,1} - k_{1,2}$ of \mathfrak{R}_Γ in GGOR parameters and got in this way the generic point c_H of this hyperplane. We then computed the Verma modules for the restricted rational Cherednik algebra \overline{H}_{c_H} . Each Verma module is of type `ModGr`, a new type for graded modules we implemented in CHAMP. The sequence of the generators defining this module is the same as the one for `RationalCherednikAlgebra`. We then apply our algorithm to this constituent closed family of local modules using the command `HeadsOfLocalModules`. We chose in this case y_1, y_2, g_2 as generators for the `ModFinder` algorithm and excluded the primes 2, 3, 7 for the random choice of a finite field specialization as these are among the bad primes. In this case the algorithm was successful and returns both the heads of the Verma modules and the decomposition matrix. Using the command `IsModuleForRRCA` we can check if a family of matrices indeed defines a module for the restricted rational Cherednik algebra—all necessary relations are checked. We see in the example that the head of $\Delta_{c_H}(\phi_{3,2})$ is indeed a module as it should be. The last command in the example computes the graded G -module structure of the simple module $L_{c_H}(\phi_{1,4})$. The result is a list of triples (d, i, m) telling that in the homogeneous component of degree d the simple KG -module indexed by i occurs with multiplicity m .

This example is the prototype showing how we can answer all of Gordon's questions by applying `HeadsOfLocalModules` to Euler families.

§8. Experimental aspects

The run time and success of the `MODFINDER` algorithm can depend heavily on the input data and on the choices made. We will therefore point out some issues we observed in experiments which are important for reproducing our main results mentioned in §10.

8.1. In table 1 we list some data concerning the computation of the Verma modules and the head of a Verma modules using our algorithm. All computations and time measurements have been performed on an Intel® Core™ i7-3930K @ 3.2GHz running the AVX version of MAGMA 2.19-8. We always work with generic GGOR parameters and use the realizations of the exceptional complex reflections groups and their representations as obtained from CHEVIE (these are also the ones used in CHAMP by default).

The columns denoted by t_Δ give the time needed for computing the X -table explained in 3.16 and the time it then takes to compute the corresponding Verma module. The column *Vars* lists the number of variables in the abstract structure of the Jacobson radical of the Verma module (note that our algorithm has to be successful to determine this number). The column *g* lists the generators we have selected for the `MODFINDER` algorithm. In the last columns denoted by $t_{\text{Hd } \Delta}$ we list the time the `MEATAXE` needed to determine the Jacobson radical of the finite field specialization of the Verma module, the time the `MODFINDER` needed and the total time (this includes for example the graded spinning algorithm to ensure that we found a submodule). This table shows us immediately how sensitive our approach is to the choices we

G	λ	$\dim \Delta$	t_Δ		$\dim \text{Hd } \Delta$	Vars	\mathbf{g}	$t_{\text{Hd } \Delta}$		
G_4	$\phi_{3,2}$	72	0.19	0.21	24	52	$\{y_2\}$	0.01	0.73	1.98
G_5	$\phi_{3,6}$	216	2.19	1.78	24	70	$\{y_2\}$	0.12	52.61	74.58
G_5	$\phi_{3,6}$	$\{y_2, x_2\}$.	12.06	33.94
G_5	$\phi_{3,6}^{(1)}$.	.	2.4	.	24	$\{y_2\}$	0.12	0.67	1.83
G_7	$\phi_{2,15}$	288	10.39	5.73	72	208	$\{y_2, y_1\}$	0.22	735.70	860.56
G_7	$\phi_{2,15}$	$\{y_2, g_1\}$.	205.01	329.81
G_{23}	$\phi_{4,4}$	480	12.15	10.27	60	759	$\{y_3, y_2\}$	3.74	40.49	72.04
G_9	$\phi_{3,4}$	576	23.56	11.16	192	491	$\{y_2\}$?	?	?
G_9	$\phi_{3,4}^{(2)}$.	.	36.72	.	90	$\{y_2\}$	1.13	4.65	11.83
G_9	$\phi_{3,4}^{(1)}$.	.	12.66	.	630	$\{y_2\}$?	?	?
G_{24}	$\phi_{3,8}^{(1)}$	1008	206.43	91.85	156	3888	$\{y_3, y_2\}$	24.22	595.72	849.51
G_{24}	$\phi_{3,10}^{(1)}$	1008	.	99.50	6	14	$\{y_3, y_2\}$	28.47	0.13	50.17

Table 1. Experimental data about the computation of the heads of Verma modules.

make throughout. Let us discuss this in more detail.

First of all, we can see that we usually work with very small \mathbf{g} . We almost never have to consider all algebra generators for the MODFINDER algorithm. For the computation of the head of the Verma module $\Delta_k(\phi_{3,6})$ for G_5 , however, we see that the selection of \mathbf{g} can be important. In this situation the choice $\mathbf{g} = \{y_2, x_2\}$ is more than twice as fast as $\{y_2\}$. Unfortunately we cannot say yet what makes one choice better than the other—we just found efficient choices by experiments.

Next, we observed that when modifying our explicit realizations of the irreducible representations of the group in such a way that some generator acts diagonally, the MODFINDER algorithm sometimes performs much faster. We denote in this table by $\lambda^{(i)}$ the representation obtained from λ by changing the basis so that the generator i of the chosen realization of the group acts diagonally. This is performed by the command `JordanizeRepresentation` in CHAMP. Comparing the computations for $\phi_{3,6}$ and $\phi_{3,6}^{(1)}$ for G_5 we see that we obtained the solution for $\phi_{3,6}^{(1)}$ around 20 times faster than for $\phi_{3,6}$. We see that the number of variables in the Jacobson radical drops from 70 to only 24 which is probably the reason for the speedup. For the example involving G_9 we are in a similar situation but see that another choice of diagonalization makes things even worse (the question marks indicate that we interrupted the computation after some time). Again we do not know how to determine in advance which choice leads to improvements.

In the example $\phi_{3,8}^{(1)}$ for G_{24} we see that even a very large number of variables (3888 in this case) do not necessarily have to be a problem. We were able to compute the head of the corresponding 1008-dimensional Verma module in just around 15 minutes. Even more fascinating is the example $\phi_{3,10}^{(1)}$ for G_{24} . Here, we finished the determination of the 1002-dimensional Jacobson radical in just 50 seconds (the MODFINDER algorithm just needed 0.13 seconds).

We see from these examples that our algorithm can be surprisingly powerful but that it is very hard to control theoretically.

8.2. So far we did not comment on other already existing algorithms to compute the heads of the Verma modules in characteristic zero. The MEATAXE might actually solve this problem in special situations. In his PhD thesis Steel [27] has developed a general characteristic zero MEATAXE which is in theory capable of computing the radical of a module over an algebra over a field of characteristic zero. This algorithm is implemented in MAGMA since 2012 and it is—to our knowledge—the only algorithm which could also be used to compute the head of Verma modules for restricted rational Cherednik algebras. We therefore have to compare our methods with this algorithm. As it is also a Las Vegas algorithm, we cannot simply test it

once for a specific problem and record the run time because it might always be the case that the randomly chosen parameters were bad. We thus have to run several tests and determine the average run time. We run each attempt with a time out τ of 900 seconds (15 minutes) for each attempt as the run time of our algorithm is always much lower. We then record the average run time of all successful approaches, and record the success rate α within the time window τ for specific problems. The results are listed in table 2.

G	λ	Tests	MAGMA Avg.	MAGMA α	CHAMP Avg.	CHAMP α
S_4	$(2, 1, 1)$	82	0.65	0.13	0.23	1.0
G_4	$\phi_{3,7}^{(1)}$	84	0.76	0.15	0.7	1.0
G_4	$\phi_{3,7}$	82	—	0.0	5.2	1.0
G_{12}	$\phi_{4,3}^{(3)}$	84	3.29	0.14	0.38	1.0
G_6	$(\phi'_{2,5})^{(2)}$	77	—	0.0	0.25	1.0
G_6	$(\phi''_{2,3})^{(2)}$	79	—	0.0	0.25	1.0
G_5	$\phi_{3,6}^{(1)}$	81	—	0.0	5.1	1.0
G_7	$(\phi'_{2,11})^{(1)}$	78	—	0.0	42.0	1.0

Table 2. Comparison of MAGMA's algorithm (left) with ours (right).

We see that our success rate is always 100% (there were only very few examples so far where our algorithm was not successful) while MAGMA's success rate is below 15%—if there is success at all. For all problems where MAGMA's algorithm did not return a result within the time window τ we also did not get a result in sporadic attempts after a couple of days. Although this does not mean that MAGMA's algorithm would not eventually solve the problem, it should be quite clear from the table that without our algorithm we would not have been able to obtain most results in §10—in particular since the modules we have to work with are much bigger than those listed in the table.

§9. The restricted rational Cherednik algebra for G_4

Using the same steps as in 7.7 we can now completely determine the representation theory of the restricted rational Cherednik algebras for G_4 for *all* parameters. We collect the results in this section. As a by-product we obtain a proof of Martino's conjecture for all parameters.

9.1. We consider the same realization of G_4 as used in CHEVIE. In CHAMP we can compute the generic Euler families with the command `EulerFamilies` and see that they are singletons. By comparing the values of the central characters on the Euler element we compute that outside of the following six hyperplanes the Euler families still remain singletons:

$$\begin{aligned}
 \text{Ia} &: k_{1,1} = 0 \\
 \text{Ib} &: k_{1,2} = 0 \\
 \text{Ic} &: k_{1,1} - k_{1,2} = 0 \\
 \text{IIa} &: k_{1,1} + k_{1,2} = 0 \\
 \text{IIb} &: 2k_{1,1} - k_{1,2} = 0 \\
 \text{IIc} &: k_{1,1} - 2k_{1,2} = 0.
 \end{aligned}$$

This computation is also performed by the command `EulerScheme` in CHAMP. As the generic Euler families are singletons, also the generic Calogero–Moser families are singletons. This implies by general theory [7] that the simple \overline{H}_k -module, where k is the generic point of $\overline{\mathfrak{R}}_\Gamma^0$ of GGOR type, are as G_4 -modules isomorphic to the regular G_4 -module. The answers to Gordon's questions are thus known in this case. The same then applies to all $k \in \overline{\mathfrak{R}}_\Gamma^0$ in an open neighborhood of the generic point. There are two questions, however, which have not been answered so far:

- (a) For precisely which $k \in \overline{\mathfrak{R}}_\Gamma^0$ is the representation theory of \overline{H}_k the same as of \overline{H}_k , i.e., which k are generic?

(b) What is the representation theory of \overline{H}_k for non-generic k ?

We can answer these questions now using CHAMP as in 7.7 by specializing to the generic point of each of the above hyperplanes and computing the heads and the decomposition of the Verma modules.

9.2. The representation theory for $\overline{H}_{k_{Ia}}$, where k_{Ia} is the rational point of the hyperplane Ia is listed in tables 3 to 5. We see from this data that the non-singleton Calogero–Moser k_{Ia} -families are

$$(29) \quad \{\phi_{1,0}, \phi_{1,8}, \phi_{2,3}\} \quad \text{and} \quad \{\phi_{2,5}, \phi_{2,1}\} .$$

The image of the hyperplane Ia under the automorphism \sharp defined in 3.9 is equal to

$$Ia^\sharp : k_{1,-1} = 0 \Leftrightarrow k_{1,2} = 0 .$$

The data computed by Chlouveraki [6] shows that the Rouquier families for parameters on Ia^\sharp (remember that we are working in $\overline{\mathfrak{M}}_\Gamma^0 \subseteq \overline{\mathfrak{M}}_\Gamma$, where $k_{1,0} = 0$) are equal to (29) so that Martino’s conjecture holds on the hyperplane Ia.

The results on the hyperplanes Ib and Ic are similar to those on Ia—the Calogero–Moser families are $\{\phi_{1,0}, \phi_{1,4}, \phi_{2,1}\}$, $\{\phi_{2,5}, \phi_{2,3}\}$ on Ib and $\{\phi_{1,4}, \phi_{1,8}, \phi_{2,5}\}$, $\{\phi_{2,3}, \phi_{2,1}\}$ on Ic—so that we skip the presentation of these results to save space (there is an automorphism responsible for this). The point is that Martino’s conjecture is still true on these hyperplanes.

λ	$\phi_{1,0}$	$\phi_{1,4}$	$\phi_{1,8}$	$\phi_{2,5}$	$\phi_{2,3}$	$\phi_{2,1}$	$\phi_{3,2}$
$\dim L_{k_{Ia}}(\lambda)$	1	24	9	16	7	8	24

Table 3. Dimension of the simple $\overline{H}_{k_{Ia}}$ -modules.

$L_{k_{Ia}}(\lambda)/\Delta_{k_{Ia}}(\lambda)$	$\phi_{1,0}$	$\phi_{1,4}$	$\phi_{1,8}$	$\phi_{2,5}$	$\phi_{2,3}$	$\phi_{2,1}$	$\phi_{3,2}$
$\phi_{1,0}$	1	.	1	.	2	.	.
$\phi_{1,4}$.	1
$\phi_{1,8}$	1	.	1	.	2	.	.
$\phi_{2,5}$.	.	.	2	.	2	.
$\phi_{2,3}$	2	.	2	.	4	.	.
$\phi_{2,1}$.	.	.	2	.	2	.
$\phi_{3,2}$	3

Table 4. Decomposition of the Verma modules of $\overline{H}_{k_{Ia}}$.

$\lambda/L_{k_{Ia}}(\lambda)$	$\phi_{1,0}$	$\phi_{1,4}$	$\phi_{1,8}$	$\phi_{2,5}$	$\phi_{2,3}$	$\phi_{2,1}$	$\phi_{3,2}$
$\phi_{1,0}$	t^0	t^4	.	.	.	t^1	t^2
$\phi_{1,4}$.	t^0	t^4	t^1	.	.	t^2
$\phi_{1,8}$.	t^8	t^0	t^3	.	.	t^2
$\phi_{2,5}$.	$t^3 + t^5$	$t^1 + t^3$	$t^0 + t^4$.	.	$t^1 + t^3$
$\phi_{2,3}$.	$t^5 + t^7$.	t^2	t^0	t^2	$t^1 + t^3$
$\phi_{2,1}$.	$t^1 + t^3$.	t^2	t^2	t^0	$t^1 + t^3$
$\phi_{3,2}$.	$t^2 + t^4 + t^6$	t^2	$t^1 + t^3$	t^1	t^1	$t^0 + t^2 + t^4$

Table 5. Structure of the simple $\overline{H}_{k_{Ia}}$ -modules as graded G_4 -modules.

9.3. The results on the hyperplane IIa are listed in tables 6 to 8. We see that the non-singleton Calogero–Moser families are equal to

$$\{\phi_{1,0}, \phi_{2,5}, \phi_{3,2}\} .$$

The image of the hyperplane IIa under the automorphism \sharp defined in 3.9 is equal to

$$IIa^\sharp : k_{1,-1} + k_{1,-2} = 0 \Leftrightarrow k_{1,2} + k_{1,1} = 0 .$$

λ	$\phi_{1,0}$	$\phi_{1,4}$	$\phi_{1,8}$	$\phi_{2,5}$	$\phi_{2,3}$	$\phi_{2,1}$	$\phi_{3,2}$
$\dim L_{k_{IIa}}(\lambda)$	3	24	24	3	24	24	18

Table 6. Dimension of the simple $\overline{H}_{k_{IIa}}$ -modules.

$L_{k_{IIa}}(\lambda)/\Delta_{k_{IIa}}(\lambda)$	$\phi_{1,0}$	$\phi_{1,4}$	$\phi_{1,8}$	$\phi_{2,5}$	$\phi_{2,3}$	$\phi_{2,1}$	$\phi_{3,2}$
$\phi_{1,0}$	1	.	.	2	.	.	3
$\phi_{1,4}$.	1
$\phi_{1,8}$.	.	1
$\phi_{2,5}$	1	.	.	2	.	.	3
$\phi_{2,3}$	2	.	.
$\phi_{2,1}$	2	.
$\phi_{3,2}$	1	.	.	2	.	.	3

Table 7. Decomposition of the Verma modules of $\overline{H}_{k_{IIa}}$.

$\lambda/L_{k_{IIa}}(\lambda)$	$\phi_{1,0}$	$\phi_{1,4}$	$\phi_{1,8}$	$\phi_{2,5}$	$\phi_{2,3}$	$\phi_{2,1}$	$\phi_{3,2}$
$\phi_{1,0}$	t^0	t^4	t^8	.	t^3	t^1	.
$\phi_{1,4}$.	t^0	t^4	t^1	t^5	t^3	.
$\phi_{1,8}$.	t^8	t^0	.	t^1	t^5	t^2
$\phi_{2,5}$.	$t^3 + t^5$	$t^1 + t^3$	t^0	$t^2 + t^4$	$t^2 + t^6$	t^3
$\phi_{2,3}$	t^1	$t^5 + t^7$	$t^3 + t^5$.	$t^0 + t^4$	$t^2 + t^4$	t^1
$\phi_{2,1}$.	$t^1 + t^3$	$t^5 + t^7$.	$t^2 + t^6$	$t^0 + t^4$	$t^1 + t^3$
$\phi_{3,2}$.	$t^2 + t^4 + t^6$	$t^2 + t^4 + t^6$.	$t^1 + t^3 + t^5$	$t^1 + t^3 + t^5$	$t^0 + t^2 + t^4$

Table 8. Structure of the simple $\overline{H}_{k_{IIa}}$ -modules as graded G_4 -modules.

The Calogero–Moser families are equal to the Rouquier families for parameters on IIa^\sharp and so Martino’s conjecture holds for such parameters. Again the results on the hyperplanes IIb and IIc are similar and Martino’s conjecture also holds for these parameters.

In total, we have answered Gordon’s questions and have proven Martino’s conjecture for all parameters of G_4 .

§10. Further results

Using CHAMP we could obtain many more results about the representation theory of restricted rational Cherednik algebras for exceptional complex reflection groups. All results are listed explicitly in [30] and we just mention here as theorems what we were able to compute as anyone can now reproduce the results using CHAMP.

10.1. Theorem. For *all* parameters for the groups

$$G_4, G_{12}, G_{22}, G_{23}, G_{24}$$

we answered all of Gordon’s questions and confirmed Martino’s conjecture for all parameters.

10.2. Theorem. For *generic* parameters for the groups

$$G_4, G_5, G_6, G_7, G_8, G_9, G_{10}, G_{12}, G_{13}, G_{14}, G_{15}, G_{20}, G_{22}, G_{23}, G_{24}$$

we answered all of Gordon’s questions and confirmed Martino’s generic parameter conjecture. The generic Calogero–Moser families are in these cases always equal to the generic Euler families.

10.3. We discussed rational Cherednik algebras for reflection groups over arbitrary fields as long as all reflections are diagonalizable and designed CHAMP to work in this generality. In [30] we computed for example the representation theory of the restricted rational Cherednik algebra attached to the general orthogonal group $GO_3(3)$ and to modular reflection representations of

some symmetric groups. These cases are not yet understood theoretically and we hope that such examples will help to develop a general theory.

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